The VL control measure for symmetric networks

Ruud Hendrickx\textsuperscript{a,}\textsuperscript{*}, Peter Borm\textsuperscript{a}, René Van den Brink\textsuperscript{b}, Guillermo Owen\textsuperscript{c}

\textsuperscript{a}CentER and Department of Econometrics and Operations Research, Tilburg University, The Netherlands
\textsuperscript{b}Department of Econometrics and Tinbergen Institute, Free University, Amsterdam, The Netherlands
\textsuperscript{c}Department of Mathematics, Naval Postgraduate School, Monterey, CA, United States

\textbf{ARTICLE INFO}

\textbf{Keywords:}
Network
Control measure
Search probabilities
Cooperative network game
Proper Shapley value
Matrix search game

\textbf{ABSTRACT}

In this paper we measure “control” of nodes in a network by solving an associated optimisation problem. We motivate this so-called VL control measure by giving an interpretation in terms of allocating resources optimally to the nodes in order to maximise some search probability. We determine the VL control measure for various classes of networks. Furthermore, we provide two game theoretic interpretations of this measure. First it turns out that the VL control measure is a particular proper Shapley value of the associated cooperative network game. Secondly, we relate the measure to optimal strategies in an associated matrix search game.

© 2008 Elsevier B.V. All rights reserved.

\section{Introduction}

Various papers have been written on measures evaluating the positions in social and economic networks. Most of these measures consider the “centrality” of the positions in symmetric (undirected) networks, cf. Freeman (1977, 1979), Monsuur and Storcken (2002) and Borgatti and Everett (2006).1 Some of these measures are obtained by applying well-known solutions to cooperative (transferable utility) games, see, e.g., Grofman and Owen (1982), Gómez et al. (2003) and Van den Brink et al. (2008).

Another stream of literature applying cooperative game theory to networks was initiated by Bienvenido and Bonacich (1992). In their approach, matchings generate value in a so-called exchange network and the positions are therefore evaluated by their “matchability”. Sozanski (2006) enriched this model by allowing for so-called exchange regimes. Since in these models it are matchings that generate value we refer to this approach as a \textit{link-based} approach.

In this paper we are interested in yet another feature of positions in networks which we call their “control” in the network. By control of a position we mean its potential ability to have influence on the whole network assuming that every position can directly communicate (or be in contact) with its direct neighbours, but needs intermediaries to reach other positions. We illustrate this interpretation by the following example.

\begin{example}
Consider a situation where there is some noxious substance hidden within each of the nodes in a network. A searcher wishes to find and uproot this substance in all nodes. He does this by distributing some fixed amount of resources (normalised to 1) among the several nodes.

It is assumed that the probability of finding and uprooting the substance which is located in a particular node equals the amount of resources placed at that node and at all adjacent nodes. The probability of finding the substance within any one node is independent of what happens in any of the other nodes. It is necessary to destroy all of the substance, because if any substance survives in any of the nodes, it can quickly reproduce and will be able to reoccupy all of the nodes.

In Example 1.1 we are interested in the allocation of the resource that maximises the probability that the noxious substance is destroyed. We define the VL control measure as the solution to this optimisation problem. We state some properties and for some special classes of networks, we provide a full description of this measure.

Also we give two game theoretic interpretations of the VL control measure: a cooperative as well as a strategic or non-cooperative interpretation. It turns out that the VL control measure coincides with a particular proper Shapley value (Vorob’ev and Liapounov, 1998, explaining the name of our control measure) applied to the associated cooperative network game introduced in Van den Brink et al. (2008)2 and that this value belongs to the core of the network.

\end{example}

1 Measures for asymmetric (directed) networks can be found in, e.g., Van den Brink and Gilles (2000) and Bonacich and Lloyd (2001).

2 In fact, looking at symmetric networks as directed networks where for each edge there are two arcs in opposite directions, this network game is the network game introduced by Van den Brink and Borm (2002) for directed networks.
game. This cooperative network game measures the power of sets or coalitions of positions in terms of its control over its neighbours. Therefore we refer to this approach as a position-based approach, as an alternative to the linked based approach of Bienenstock and Bonacich (1992) and Sozanski (2006) referred to above.³

Besides the argument of giving an optimal resource allocation in networks as in Example 1.1, we provide an alternative strategic interpretation for the VL control measure related to the setting of the next example.

Example 1.2. Consider a situation where a searcher wants to find a single terrorist who hides himself in one particular node of the network. The searcher finds the terrorist if they are in the same node or in adjacent nodes. The objective of the searcher is to maximise the probability of finding this terrorist before he carries out an attack. Therefore we refer to this approach as a position-based approach to measure power in asymmetric networks is developed. Since it is necessary to locate the substance in all nodes simultaneously, we find that the searcher wants to maximise the joint probability \( \prod_{i \in N} y_i = \sum_{j \in R_c(i)} \sum_{j' \in R_c(j)} x_{ij} \) subject to the constraint that the quantities \( x_{ij} \) must be non-negative and add to 1. In this way we are faced with the maximisation problem given by

\[
\begin{align*}
\text{max} & \quad \prod_{i \in N} y_i \\
\text{subject to} & \quad y_i = \sum_{j \in R_c(i)} x_j \quad \text{for all } i \in N, \\
& \quad \sum_{j \in N} x_{ij} = 1, \\
& \quad x_{ij} \geq 0 \quad \text{for all } j \in N.
\end{align*}
\]

The optimal value of (2.1) for a network \( G \in G^N \) is denoted by \( f(G) \) and the set of corresponding solutions for \( x \) is denoted by \( VL(G) \subset S^N \), where \( S^N = \{ x \in R^N \mid x_i \geq 0, \sum_{i \in N} x_i = 1 \} \) is the unit simplex in \( R^N \). We refer to any vector in \( VL(G) \) as a VL control measure of \( G \).

Freeman (1977) distinguishes between three types of centrality measures for networks: point measures (assigning values to nodes in networks), normalised point measures, and network measures (assigning values to networks). The last type is meant to compare two networks with the same number of nodes in terms of the spread of centrality within the network. Note that in terms of centrality a star and the complete graph can be seen as two extreme cases, see also Mackenzie (1966). If we extend Freeman’s classification to the current setting, the VL control measure can be seen as a normalised point measure. In computing the VL control measure, we obtain the maximum of the objective function in (2.1), which one can use as a network measure representing the total control (of the searcher) in the network. However, in this paper we focus on normalised control power measure in \( VL(G) \).

Throughout this section, we reserve the vectors \( x \) and \( y \) for the role they take in program (2.1). If \( n = 1 \) or \( n = 2 \), then clearly every \( x \in S^N \) is optimal (remember that we only consider connected networks), so from now on, assume

\[ n \geq 3. \]

Note that by taking \( x = (1/n, \ldots, 1/n) \), we immediately conclude that \( f(G) \) is positive.

In general, solving (2.1) can be quite complicated, depending on the form of the network \( G \). One obvious first step in finding the solution is presented in the following lemma.

Lemma 2.1. Let \( i \in N \) be a pending node in a network \( G \). Then it can never be optimal to have \( x_i > 0 \).

Proof. The single node \( j \in R_c(i) \setminus \{ i \} \) must have at least two neighbours (besides itself), because \( G \) is connected and \( n \geq 3 \). Let \( x \in S^N \) be such that \( x_i > 0 \). Then \( x' \) given by \( x'_j = x_j + x_i \) and \( x'_k = 0 \) for all \( k \in N \setminus \{ i, j \} \), gives an improvement, since it strictly increases \( y_k \) for all \( k \in R_c(j) \setminus \{ i, j \} \), while otherwise leaving \( y \) unchanged.

We are going to solve the maximisation program (2.1) for some special classes of networks. Clearly, the value \( f(G) \) of the program (2.1) is positive and cannot be greater than one. This upper bound is obtained for networks that have a central node that is directly connected to all other nodes.

Proposition 2.2. For every \( G \in G^N \) it holds that \( f(G) \leq 1 \). Moreover, \( f(G) = 1 \) if and only if there exists a node \( i \in N \) such that \( R_c(i) = N \).

Proof. It is obvious that \( f(G) \leq 1 \). To prove the second statement, first assume that \( R_c(i) = N \) for some \( i \in N \). Take \( x \in S^N \) such that \( x_i = 1 \) and \( x_j = 0 \) for all \( j \in N \setminus \{ i \} \). Since \( i \in R_c(j) \) for all \( j \in N \), \( y_j = 1 \) for all \( j \in N \) and \( f(G) = 1 \).

Next, suppose that \( G \) is such that there is no \( i \in N \) such that \( R_c(i) = N \). Let \( x \in S^N \) and let \( i \in N \) be such that \( y_i > 0 \). Take \( j \in N \setminus R_c(i) \), which is possible by assumption. Then \( y_j \leq 1 - x_i < 1 \) and consequently,
the objective value is smaller than 1. Since this holds for all \( x \in S^N \), \( f(G) < 1 \). Hence, \( f(G) = 1 \) if and only if there exists a node \( i \in N \) such that \( R_G(i) = N \). □

In particular the maximal value \( f(G) \) is obtained for a star network, i.e., a network \( G \) such that there is an \( i \in N \) with \( R_G(i) = N \) and \( R_G(j) = \{i,j\} \) for all \( j \in N \setminus \{i\} \). Also, \( f(G) = 1 \) if \( G \) is the complete network: \( \{i,j\} | i, j \in N, i \neq j \).

**Theorem 2.3.** If \( G \in G^N \) is a star network with central position \( i \in N \), then \( f(G) = 1 \) and \( VL(G) = |x| \) with \( x \in \mathbb{R}^N \) given by \( x_i = 1 \) and \( x_j = 0 \) for all \( j \in N \setminus \{i\} \). If \( G \in G^N \) is the complete network, then \( f(G) = 1 \) and for any \( x \in S^N \) we have \( x \in VL(G) \).

**Proof.** The two statements immediately follow from the proof of **Proposition 2.2.** □

Next, we consider the case of a path network, defined as a network \( G \) consisting of the edges \( \{i, i+1\} | i \in \{1, n-1\} \), as depicted in Fig. 1. Using **Lemma 2.1**, an optimal \( x \) must satisfy \( x_1 = x_n = 0 \). For all other \( i \), the variable \( x_i \) contributes to three of the coordinates of \( y \), and we conclude that in the optimum we must have

\[
\sum_{i \in N} y_i = 3. \tag{2.2}
\]

Hence, the arithmetic mean of the variables \( y_i \) must be equal to \( 3/n \). By a well-known inequality, the geometric mean cannot be greater than this. But our objective function is precisely the \( n \)-th power of the geometric mean, so for our program we obtain the theoretical upper bound

\[
\max \prod_{i \in N} y_i \leq \left( \frac{3}{n} \right)^n. \tag{2.3}
\]

Note, however, that this is an upper bound, not necessarily the maximum. To obtain this upper bound, all of the variables \( y_i \) (not just their arithmetic mean) must be equal to \( 3/n \). This cannot always be done, as is easily checked by considering the case \( n = 4 \). For \( n = 6 \), however, the assignment

\[
x = \begin{pmatrix} 0, & 1, & 0, & 0, & \frac{1}{2}, & 0 \end{pmatrix}
\]

will clearly do the job. In fact, this can be done whenever \( n \) is a multiple of 3. For, in such a case, let \( x \) be given by

\[
x_i = \begin{cases} \frac{3}{n} & \text{if } i \equiv 2, \\ 0 & \text{if } i \equiv 0, 1, \end{cases}
\]

where all equivalences \( \equiv \) are mod 3. It is easily checked that this will give the desired value \( (3/n) \) for each coordinate of \( y \). If \( n \) is not a multiple of 3, the situation is slightly more complicated.

---

4 Having **Proposition 2.2**, one obvious difference with Bienenstock and Bonachich (1992) becomes apparent. Consider the exchange networks (a) and (f) in their Fig. 1. In both networks, position \( B \) has full control (i.e., assigning 1 to node \( B \) is the solution of (2.1)). When one considers matchings, however, there is a clear difference between the two networks, because in network (f), positions \( D1 \) and \( D2 \) can generate value on their own. **Proposition 2.2** also contrasts with Mackenzie (1966), who argues that the maximum difference in centrality power between two nodes in a network should occur in a star, while the minimum difference should occur in the complete network. In terms of control, these two networks yield the same result.
Next, we consider the case $n = 2$, so let $n = 3q + 2$ with $q \in \mathbb{N}$. Then the original program (2.1) can be relaxed to:

$$\max \prod_{i \in N} y_i$$

such that

$$\sum_{i \in N} y_i = 2,$$

$$\sum_{i \in N, i=1,2} y_i = 1,$$

$$y_i \geq 0 \quad \text{for all } i \in N.$$

This maximisation problem has the following solution:

$$y_i = \begin{cases} \frac{1}{q} + \frac{1}{q} & \text{if } i = 1, 2, \\ \frac{1}{q} & \text{if } i = 0. \end{cases}$$

Again, we solve $M_\alpha x = y$, which with Lemma 2.1 results in the following unique solution:

$$x = \frac{1}{q(q+1)}(0, q, 0, 0, 0, q - 1, 0, 2, 0 \ldots, 0, q - 1, 0, 1, 0, q, 0).$$

The solution of (2.1) in case $G$ is a path network is summarised in the following theorem.

**Theorem 2.4.** Let $G \in G^N$ be a path network with $n = 3q + r$, $q \in \mathbb{N}$, $r < 3$. Then the optimal value $f(G)$ of (2.1) and the unique solution $VL(G) = \{x\}$ with $x \in \mathbb{R}^N$ are given by the following three cases, depending on $r$:

(a) $r = 0$: $f(G) = \left(\frac{1}{3}\right)^n$ and

$$x = \left(0, \frac{1}{q}, 0, 0, 0, \frac{1}{q}, 0, 0, \ldots, 0, \frac{1}{q}, 0\right).$$

(b) $r = 1$: $f(G) = \left(\frac{1}{q}^{q+1}\right)\left(\frac{1}{3}\right)^q$ and

$$x = \frac{1}{q(q+1)}(0, q, 0, 1, q - 1, 2, 0, q - 2, 0, \ldots, 0, q - 1, 0, 1, q, 0).$$

(c) $r = 2$: $f(G) = \left(\frac{1}{q+1}\right)^{2q+1}\left(\frac{1}{3}\right)^q$ and

$$x = \frac{1}{q(q+1)}(0, q, 0, 1, q - 1, 0, 2, q - 2, 0, \ldots, 0, q, 0).$$

One natural question is whether for a given set $N$ of nodes, the minimum value of $f(G)$ over all networks $G$ on $N$ is obtained when $G$ is a path as is, e.g., the case with the concept of centrality in Gómez et al. (2003). This turns out not to be true, as is shown in Example 2.1.

**Example 2.1.** Consider $N = \{1, \ldots, 7\}$. For the path network $G'$ on $N$, we use Theorem 2.4 to get

$$f(G') = \left(\frac{1}{3}\right)^3 \left(\frac{1}{2}\right)^4 = \frac{1}{432}.$$

Next, consider the network $G$ depicted in Fig. 2.

As can be easily seen, the maximum of (2.1) for this network is obtained in $x = (0, 1/3, 0, 1/3, 0, 1/3, 0)$, so

$$f(G) = \left(\frac{1}{3}\right)^6 \cdot 1 = \frac{1}{729}.$$

Hence, $f(G') > f(G)$.

A final interesting type of network that we consider is a cycle, defined as a network $G$ consisting of the path and the extra link $(n, 1)$, as depicted for $n = 5$ in Fig. 3. Again, for all $x \in \mathbb{R}^N$, we have (2.2) for the corresponding optimal $y$. Contrary to the path case, however, we can always and easily find an $x$ such that $y_i = 3/n$ for all $i \in N$ and the upper bound in (2.3) is reached.

**Theorem 2.5.** Let $G \in G^N$ be a cycle. Then $f(G) = (3/n)^n$ and for $x \in \mathbb{R}^N$ with $x_i = 1/n$ for all $i \in N$ we have $x \in VL(G)$.

3. A relation with cooperative network games

A cooperative game with transferable utility (or simply a TU game) is a pair $(N, v)$, often just represented by $v$, with player set $N = \{1, \ldots, n\}$ and characteristic function $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. The real number $v(S) \in \mathbb{R}$ represents the worth that the players in coalition $S \subseteq N$ can jointly guarantee if they cooperate.

In the literature, cooperative TU games have been applied to measure power in networks by associating to every network a TU game in which the nodes are identified as players and the characteristic function assigns to every coalition of nodes a worth that can be seen as a measure of the power of that coalition. Power of individual positions then can subsequently be measured by applying general TU solution concepts (such as the core and the Shapley value, which we discuss later in this section) to the associated games. We distinguish two approaches of measuring power of coalitions of nodes. In link-based approaches, such as developed in, e.g., Bienenstock and Bonacich (1992) and Sozanski (2006), power is measured by considering the matchings that can be made within coalitions. On the other hand, in position-based approaches, such as in, e.g., Van den Brink and Gilles (2000) and Van den Brink and Borm (2002), power is measured by considering to which positions each coalition is linked.

In our basic interpretational settings described in Examples 1.1 and 1.2, we consider the power of a position in terms of its control over its neighbours. Hence, we follow the position-based approach for symmetric networks as developed in Van den Brink et al. (2008) who associate to every network $(N, G)$ the so-called (conservative) network (power) game $(N, v_G)$ with characteristic function $v_G(S) = |\{i \in R_G(S) \mid R_G(i) \subseteq S\}|$.

---

5. Note that in this example, the VI-control measure yields the same allocation as the matching-based result in Bienenstock and Bonacich (1992), network 1d.

6. Van den Brink et al. (2008) allow for loops to be part of the network and consequently have a different notion of neighbour.
for all $S \subset N$. That is, the worth of a coalition $S$ of players equals the number of neighbours of $S$ that have no neighbours outside $S$. The idea behind this network game is that it in some sense measures the “power” of the nodes in the network. In Van den Brink et al. (2008), who want to measure centrality power, the interpretation of the network game is that the worth of a coalition $S$ represents the number of neighbours of $S$ that have no neighbours outside $S$, and thus for communication fully depend on $S$. However, in our control setting we reinterpret this network game by noting that the amount of resource allocated to the nodes in $S$ fully determines the probability of finding and uprooting the noxious substance in the neighbours of $S$ that have no neighbours outside $S$, and thus the nodes in $S$ fully control the search probabilities in these nodes.

Because in our framework $i \in R_G(i)$ for all $i \in N$, one finds that $v_G(S) = |\{i \in S | R_G(i) \subset S\}|$. (3.1)

**Example 3.1.** To illustrate the network game, consider again the network $G$ in Fig. 2. Since each one-player coalition has neighbours outside that coalition, $v_G(\emptyset) = 0$ for all $i \in N$. Similarly, most two-player coalitions have worth 0, except $\{1, 2\}, \{4, 5\}$ and $\{6, 7\}$, which have worth 1 because the pending nodes $1, 5$ and $7$ have no neighbours outside their respective coalition. The worth of a coalition of a certain size very much depends on how this coalition is connected, as can be seen from $v_G(\{1, 2, 3, 4, 7\}) = 2$ and $v_G(\{1, 2, 3, 4, 5\}) = 4$. Of course $v_G(N) = n = 7$.

The network game $v_G$ can be nicely expressed as a linear combination of so-called unanimity games. The class of unanimity games constitutes a basis for the space of all TU games. Every TU game $v$ can be expressed as a linear combination of unanimity games in a unique way (Harsanyi, 1959):

$$v = \sum_{T \subset N, T \neq \emptyset} \Delta_v(T)u_T,$$

where $u_T$ is the unanimity game of coalition $T \subset N$, $T \neq \emptyset$, defined by $u_T(S) = 1$ if $T \subset S$ and $u_T(S) = 0$ otherwise, and the coefficients are the dividends, defined by $\Delta_v(S) = v(S)$ if $|S| = 1$ and recursively $\Delta_v(S) = v(S) - \sum_{T \subset S, T \neq \emptyset} \Delta_v(T)$ if $|S| \geq 2$. So, for every $S \subset N$ its worth can be written as $\nu(S) = \sum_{T \subset N, T \neq \emptyset} \Delta_v(T)u_T(S) = \sum_{T \subset S, T \neq \emptyset} \Delta_v(T)$. So the worth of any coalition equals the sum of the dividends of all its subcoalitions. The dividend of coalition $S$ in a network game $v_G$ equals

$$\Delta_v(S) = |\{i \in S | R_G(i) = S\}|$$

for all $S \subset N$, $S \neq \emptyset$. For $|S| = 1$ this follows immediately from (3.1). For larger $S$, it follows by induction (assuming that (3.2) holds for all $T \subset S$):

$$\Delta_v(S) = v(S) - \sum_{|T \subset S, T \neq \emptyset} \Delta_v(T)u_T = |\{i \in S | R_G(i) \subset S\}| - \sum_{T \subset S, T \neq \emptyset} |\{i \in T | R_G(i)\}| = |\{i \in S | R_G(i) = S\}|.$$ Using (3.2), Van den Brink et al. (2008) show that

$$v_G = \sum_{i \in N} u_{R_G(i)}.$$ Hence every network game is totally positive, meaning that it can be expressed as a non-negative linear combination of unanimity games, i.e., all dividends are non-negative (see, e.g., Vasiliev, 1975; Hammer et al., 1977). This implies in particular that a network game is convex meaning that $\nu(S) + \nu(T) \leq \nu(S \cup T) + \nu(S \cap T)$ for all $S, T \subset N$.

An allocation for a TU game $v$ on $N$ is a vector $x \in \mathbb{R}^N$ with $x_i$ representing the payoff to player $i$ in this game. The core of a game consists of those allocations that are efficient and coalitionally rational:

$$C(v) = \left\{ x \in \mathbb{R}^N | \sum_{i \in S} x_i = v(N), \forall S \subset N : \sum_{i \in S} x_i \geq v(S) \right\}.$$ The Shapley value (cf. Shapley, 1953b) of a game divides for each coalition $S \subset N$ the dividend $\Delta_v(S)$ equally over all players in $S$:

$$\phi_i^v = \sum_{S \subset N} \Delta_v(S) / |S|.$$ It follows from convexity that each network game has a non-empty core, in particular the Shapley value is an element of the core.

Instead of distributing the dividends equally among the players in the corresponding coalition, given a system of positive weights $\omega \in \mathbb{R}^N$ with $\omega_0 > 0$ for all $i \in N$, the weighted Shapley value (cf. Shapley, 1953a) $\Phi^v(\omega)$ of game $v$ allocates the dividends proportional to the weights: $\Phi^v(\omega) = \sum_{i \in S} (\omega_i / \omega(S)) \Delta_v(S)$ for all $i \in N$, where $\omega(S) = \sum_{i \in S} \omega_i$. Clearly, we obtain the Shapley value by taking equal weights. Of course, given a weight system $\omega$, we get the same weighted Shapley value if we normalise the weights such that they add to one. Hence, we only consider weight systems taken from the interior of the unit simplex $\mathbb{S}^N$.

Vorob’ev and Liapounov (1998) extend the definition of $\Phi^v$ to the whole unit simplex (and not just its interior) by taking its closure. The mapping $\Phi^v : \mathbb{S}^N \rightarrow \mathbb{R}^N$ that assigns to every weight system $\omega \in \mathbb{S}^N$ the weighted Shapley value $\Phi^v(\omega)$ of game $v$ is called the Shapley mapping of game $v$. For a game $v$ with non-negative dividends and $\nu(N) = 1$, a fixed point of the Shapley mapping, i.e., a vector $\omega_0 \in \mathbb{S}^N$ such that $\Phi^v(\omega_0) = \omega$, is called a proper Shapley value (cf. Vorob’ev and Liapounov, 1998).

Vorob’ev and Liapounov (1998) show that the maximisation problem

$$\max_{x \in \partial \mathbb{S}^N} \prod_{S \subset N} \left( \sum_{j \in S} x_j \right) / \Delta_v(S)$$

yields a proper Shapley value of $v$. We refer to a solution of this maximisation problem as a VL value of $v$. For any game $v$ with non-negative dividends and $\nu(N) > 0$, we define $\nu(N)$ times a VL value of the equivalent 1-normalisation of $v$ to be a VL value of $v$ itself.

Whereas Van den Brink et al. (2008) have shown that the Shapley value of the network game can be interpreted as a centrality power measure respecting the power dependence theory of Emerson (1962), surprisingly it turns out that applying the concept of VL value to the network game yields the VL control measure of the underlying network.

---

7 This follows straightforwardly since $v_G = \sum_{T \subset N, T \neq \emptyset} \Delta_v(T)u_T = \sum_{T \subset N, T \neq \emptyset} \Delta_v(T)u_T$.

8 The 1-normalisation $w$ of a game $v$ with $\nu(N) \neq 0$ is defined by $w(S) = (\nu(S) / \nu(N))$ for all $S \subset N$. 

---
For a network $G \in \mathcal{G}^N$, using (3.2), VL values are generated by the maximisation problem

$$\max_{x \in \mathcal{S}^N} \prod_{i \in N} \sum_{j \in R(i)} x_j$$

This optimisation problem can be rewritten as

$$\max_{x \in \mathcal{S}^N} \prod_{i \in N} \sum_{j \in R(i)} x_j,$$

and thus is equivalent to the maximisation problem (2.1) which defined our VL control measure. Therefore, any VL control measure of $G$ can be seen as a VL value of the corresponding cooperative network game $\nu_G$. Moreover, since all weighted Shapley values belong to the core of a totally positive game, any VL control measure, being a particular weighted Shapley value, belongs to the core of the cooperative network game.

We summarise our findings of this section in the following theorem.

**Theorem 3.1.** For every $G \in \mathcal{G}^N$, each element of $\nu_G$ is a VL value of the cooperative network game $\nu_G$. Moreover, $\nu_G(\nu_G) \subseteq \mathcal{C}(\nu_G)$.

4. A relation with a matrix search game

In this section we consider the maximisation program (2.1) from the non-cooperative perspective of finding a terrorist, as described in Example 1.2.

For a network $G \in \mathcal{G}^N$, we denote by $A_G$ the $N \times N$ adjacency matrix of $G$, so $A_G$ is a symmetric matrix with

$$A_G^G = \begin{cases} 1 & \text{if } i = j \text{ or } (i, j) \in E, \\ 0 & \text{if } i \neq j \text{ and } (i, j) \notin E. \end{cases}$$

This matrix $A_G$ gives rise to a search matrix game, where both players’ mixed strategy space is $\mathcal{S}^N$. This matrix game describes the terrorist search. Both player 1 (the searcher) and player 2 (the terrorist) place probabilities on the nodes. The searcher does this in such a way that he maximises his probability (given the terrorist’s strategy choice) of ending up in a “1”, i.e., of finishing at the terrorist’s node or an adjacent node. Hence, the searcher aims to minimise the probability of the terrorist repositioning $y_i$ rather than the product, as he did in Example 1.1. The terrorist’s objective is to evade the searcher and maximise the probability of ending up in a “0”.

Player 1’s set of optimal strategies in the matrix game $A_G$ is given by

$$O_1(A_G^G) = \{x \in \mathcal{S}^N \mid \min_{x \in \mathcal{S}^N} x^\top A_G^G z = \max_{x \in \mathcal{S}^N} \min_{x \in \mathcal{S}^N} (x')^\top A_G^G z \}.$$

In this section, we study the relationship between the VL control measure for the special types of networks considered in Section 2 and player 1’s optimal strategies in the matrix games arising from the corresponding adjacency matrices.

If $i \in N$ is such that $R(i) = N$, then the row corresponding to $i$ in the adjacency matrix $A^G$ contains only ones. It is immediately clear that then the vector $x \in \mathcal{S}^N$ with $x_i = 1$ and $x_j = 0$ for all $j \in N \setminus \{i\}$ is an optimal strategy for player 1 in the matrix game $A_G$. Note that this is a VL value of the corresponding network game, see Theorem 2.3.

**Theorem 4.1.** Let $G \in \mathcal{G}^N$ be a star or the complete network. Then for $x \in \nu_G$ as given in Theorem 2.3 we have $x \in O_1(A_G^G)$.

In particular, for a star network and a complete network, finding an $x \in \mathcal{S}^N$ that maximises (2.1) boils down to finding an optimal strategy for player 1 in the associated matrix game $A^G$. Also for cycle networks, the VL control measure turns out to be an optimal strategy, see Theorem 2.5.

**Theorem 4.2.** Let $G \in \mathcal{G}^N$ be a cycle network. Then for $x \in \nu_G$ as given in Theorem 2.5 we have $x \in O_1(A_G^G)$.

For path networks, the VL control measure is also an optimal strategy for player 1.

**Theorem 4.3.** Let $G \in \mathcal{G}^N$ be a path network. Then for $x \in \nu_G$ as given in Theorem 2.4 we have $x \in O_1(A_G^G)$.

**Proof.** Let $n = 3q + r$ with $q \in \mathbb{N}$, $r < 3$. We first prove the statement for $r = 1$. Using Theorem 2.4, it suffices to show that for all $x' \in \mathcal{S}^N$,

$$\min_{i \in N} x'_i A_G^G \leq \frac{1}{q + 1}.$$

But this holds, since

$$\min_{i \in N} x'_i A_G^G \leq \frac{1}{q + 1} \sum_{i \in N} x'_i = \frac{1}{q + 1}.$$

where (4.1) follows from the observation that the minimum of the numbers $\sum_{j \in R(i)} x'_j$ cannot exceed their average.

For $r = 0$, we use the same construction to obtain

$$\min_{i \in N} x'_i A_G^G \leq \min_{x_{i, N} \in \mathcal{S}^N} \sum_{j \in R(i)} x'_j \leq \frac{1}{q} \sum_{j \in N} x'_j = \frac{1}{q} \frac{1}{q + 1} = \frac{1}{q + 1}.$$

and for $r = 2$ we have

$$\min_{i \in N} x'_i A_G^G \leq \min_{x_{i, N} \in \mathcal{S}^N} \sum_{j \in R(i)} x'_j \leq \frac{1}{2(q + 1)} \sum_{j \in N} x'_j = \frac{1}{q + 1}.$$

_Apparently, in case the network is a star, path or cycle, the optimal searching strategies for finding the noxious substance are also optimal for finding the terrorist (but not necessarily vice versa). In general, however, the shape of the network does make a difference, as is shown in our final example.

**Example 4.1.** Consider the network $G$ depicted in Fig. 4. Then $x = (0, 2, 3, 0, 3, 0, 3, 0) \in \nu_G$. This vector, however, is not an optimal strategy for player 1 in the matrix search game $A_G$, since

$$\min_{i \in N} (x')^\top A_G^G \leq \frac{1}{3},$$

while for $x' = (0, 1/2, 0, 1/2, 0)$ we have

$$\min_{i \in N} (x')^\top A_G^G = \frac{1}{2}.$$
5. Concluding remarks

In this paper we measured “control” in a network by finding the solution of a maximisation problem that was interpreted as maximising some probability of finding and uprooting a noxious substance in a network. We provided properties and full descriptions of this measure for special classes of networks and gave both a cooperative and strategic (non-cooperative) foundation.

Whereas we saw that any VL control measure can be obtained as a VL value of the corresponding cooperative network game, Van den Brink et al. (2008) considered the Shapley value of this game which can be seen as a centrality measure. Since any measure of centrality is bound to give position 3 in the network of Fig. 1 the highest value, the VL control measure cannot be considered a centrality measure since it assigns value zero to position 3. The reason in terms of control is that to reach the pending positions, the positions 2 and 4 should get positive value, but then it is superfluous to assign positive value to position 3.

The results in this paper yield various questions for future research. First, the VL control measure can be studied for a wider variety of network shapes than those presented in this paper. Moreover, one might extend the analysis to heterogenous networks, in which the edges have different strengths. In addition, one might allow for the case of partially asymmetric networks, and study control based on the VL value of the cooperative games introduced in Van den Brink and Gilles (2000) and Van den Brink and Borm (2002).

Acknowledgement

This author acknowledges financial support from the Netherlands Organisation for Scientific Research (NWO).

References


