The Axioms of Quantity and the Theory of Measurement

Translated from Part I of Otto Hölder's German Text
"Die Axiome der Quantität und die Lehre vom Mass"*

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INTRODUCTION

By Joel Michell

Otto Ludwig Hölder (born, Stuttgart, 1859; died, Leipzig, 1937) was professor of mathematics at the University of Leipzig (Hölder, 1972) when he published the paper here translated into English. Today, Hölder is remembered in mathematics for a variety of results, but he is of significance to readers of this journal because of Hölder's theorem. This theorem occupies a central place in modern measurement theory. Indeed, Luce reports that it has been said that "representational measurement theory is nothing but applications of Hölder's theorem" (1994, p. 4). Furthermore, its applications have produced results with revolutionary implications for quantification in psychology and cognate sciences. In particular, Krantz, Luce, Suppes, and Tversky (1971) "unified measurement theory by reducing proofs of most representation theorems to Hölder's theorem" (Luce, Krantz, Suppes, & Tversky, 1990, p. 2) and, more recently, Luce et al. (1990) used Hölder's theorem in proofs relating to nonadditive structures.

These applications obscure the fact that Hölder's concerns were not quite those of modern theorists. His paper is a watershed in measurement theory, dividing the classical (stretching from Euclid) and the modern (stretching to Luce et al., 1990) eras. His concerns belonged, in part, to the classical era. He axiomatised the classical concept of quantity (using Dedekind's concept of continuity) in such a way that ratios of magnitudes (as understood in Book V of Euclid’s Elements) could be expressed as positive real numbers (as intimated by Newton). Importantly, he achieved this result with a clarity that was not attained by others also working within the classical framework (e.g., Helmholtz, 1878; Frege, 1903; and Whitehead & Russell, 1913).

However, other of his concerns were more modern. In Part II of his paper (a translation of which is to be published later), he showed that axioms for an apparently nonadditive structure, stretches of a straight line, entail that linear distances satisfy the axioms for quantity given in Part I. His concern to relate nonadditive structures to quantitative ones anticipates a distinctly modern interest and in doing this one of his axioms was not unrelated to the Thomson condition of conjoint measurement theory. That these results might have found direct psychophysical application is obvious and had the founder of modern experimental psychology, Wilhelm Wundt, who had established his famous laboratory at the same university, known as much of Hölder's work as Hölder knew of his (see Hölder, 1900), the development of psychological measurement might have been transformed. That Hölder's contribution eventually found its pivotal place in modern measurement theory is testimony to its enduring importance. That it was neglected within psychology until the middle of this century was psychology's loss. Contemporaneous recognition of his work within experimental psychology may have averted the criticisms of the Ferguson Committee (Ferguson, et al., 1940) and the crisis in psychological measurement theory that ensued, thereby making unnecessary Stevens' (1951) attempt at a rational reconstruction of psychological measurement and the confusions that it delivered upon the discipline.

Hölder's axioms of quantity (given in §1) have been criticised by Nagel (1932), Suppes (1951), and Luce et al. (1990), not because of any formal inadequacy leading to his main results, but because they do not match the kind of empirical structures of interest to these authors. I will not
enter this debate here, but its existence indicates another difference between the work of Hölder and modern psychological work on measurement theory. Hölder was not especially interested in more or less directly observable empirical structures (say, of the kind discussed in Suppes & Zinnes (1963) or Krantz et al. (1971)). His interest was in what he termed absolute, continuous quantity, which is an essentially theoretical concept invoked to explain observations, but one not, itself, always open to direct verification. It is the kind of concept that underpins most quantitative theorising in science and, also, within psychology. If, in proposing a theory, a scientist takes a certain set of attributes to be quantitative, then it is usually quantity in something like Hölder’s sense that is presumed. If, furthermore, the scientist wished to experimentally test the hypothesis that the attributes proposed really are quantitative, then a consideration of observable structures of the kind discussed by these authors would be relevant if the scientist took the extra step of augmenting the theory by linking the hypothesized quantities to observable structures of that sort.

Failure to appreciate this distinction has led, I believe, to a possible misunderstanding of Hölder’s project by modern measurement theorists. They consistently interpret his axioms as a contribution to the theory of extensive measurement (e.g., Suppes, 1951; Krantz et al., 1971; Narens, 1985; Niederee, 1992; Savage & Ehrlich, 1992). Now, if by extensive measurement is meant measurement via a directly observable concatenation operation, then this interpretation of Hölder’s project is questionable. Hölder does not use this term to describe his set of axioms (and the term was familiar then), the intensive/extensive distinction originating in the Middle Ages and being an important feature of the then influential Kantian understanding of measurement). Furthermore, it is by no means clear that he intended the additive operation within his axioms to be understood as a directly observable one. This interpretation of Hölder’s axioms caused philosophers from the start (e.g., Nagel, 1932) to attempt to improve upon him by shifting away from his essentially theoretical concept of quantity, to more directly testable, but less concise, axioms. Perhaps the most significant shift away from his axioms is the latter-day preference for Archimedean structures over continuous (or Dedekind complete) ones. Whatever the philosophical merits of this shift, Hölder is correct in regarding continuity as a feature of the scientific concept of quantity, both then and now, and it is not obvious that merely Archimedean structures can do the theoretical job required.

Naturally, Hölder was well aware of the distinction, but because, in his paper, he was unfolding the conceptual commitments of the scientific concept of continuous quantity, he confined his remarks about Archimedean structures to a series of footnotes (see his footnotes 5, 8, 9, 21, 22, & 31; throughout the paper, a boldface numeral indicates a translators’ note, and a lightface numeral indicates an author’s note. These are presented as endnotes), except to show that the Archimedean property follows from his preferred axioms (see §4). It is in these footnotes that the gist of his now well-known eponymous theorem is implied, although never formally stated as such. For Hölder, the key theorem of his paper was that given in §10, that for each ratio of magnitudes there exists a positive real number, for this result ensures the measurability of magnitudes of a continuous quantity and allows the theory of proportions given by Euclid to be derived more lucidly than otherwise.

So the account of measurement theory given by Hölder is not intended as an account of how measurement is attained in practice, as, for example, Campbell’s (1920) was. The question that interested Campbell and most subsequent measurement theorists was how measurable empirical structures can be defined via observation and experiment. Hölder showed little interest in this. The question that interested him was this: In hypothesising that an attribute is scientifically measurable, what kind of structure must it have in order that ratios of its magnitudes can be expressed as positive real numbers? Bunge (1973) has drawn attention to this, but mistakenly attributes the misinterpretation to the translation of Hölder’s term, Mass, as measurement. That Mass is not easily translated, I acknowledge (see Translators’ Notes, 1), but Bunge’s suggestion fails to recognise that measurement is ambiguous (and now includes what Hölder meant by Mass) and, more importantly, that Hölder’s question mattered less within the modern paradigm than it had within the classical. Thus, Hölder’s contribution reminds us that there are two sides to measurement theory: one side (emphasised in the modern era) at the interface with experimental science, the other side (emphasised in the classical) at the interface with quantitative theory. Concentration upon the former has caused the concept of quantity to be somewhat neglected recently and the point of Hölder’s work to be misunderstood. However, the concept of quantity cannot be permanently ignored and recent work by Luce (1986, 1987) on automorphic translations within relational structures points in the direction of territory explored by Hölder. There is an obvious conceptual link between translations upon ratio scalable structures and Hölder’s ratios of magnitudes.

There is another respect in which Hölder has, perhaps, been misinterpreted and this has to do with whether or not his view of measurement was a representational one (see Michell, 1993). It is far from clear from the contents of this paper exactly what view Hölder took of the real numbers and, especially, of their conceptual link with ratios of magnitudes, but some signs are suggestive. This and other works by him (e.g., Hölder, 1900, 1924) reveal a deep interest in the foundations of mathematics, and his interest in empirical and constructivist conceptions within mathematics is also evident. Hence, over against the now familiar
representationalist interpretation of his paper, other options may be considered. One suggestive feature is that he commences his paper by contrasting two approaches to the axiomatisation of arithmetic, one of these being via axioms of quantity. In light of this, it is at least possible that Hölder understood his axioms as sustaining real numbers via ratios of magnitudes, rather than merely establishing a correspondence between real numbers and such ratios, construed as logically independent concepts. This possibility is strengthened by his footnote 28, where, building upon both Euclid and Dedekind, he defines identity conditions for such ratios via real numbers (or cuts) which, of course, reduce to sets of inequalities between multiples of magnitudes. This is a curious move if he did not see them as part of the same conceptual package. Clearly, we must await further research on his philosophical views before we can confidently affirm a representationalist interpretation as his. From a purely mathematical point of view there is nothing new now new in Hölder’s paper. Its interest to the modern reader is historical and philosophical. Historically, it is revealing to see the level of mathematical and conceptual sophistication to which Hölder raised measurement theory. Between then and Suppes (1951), little in the English-language measurement literature matches him. Those who wrote in that period (e.g., Russell, 1903; Campbell, 1920; Nagel, 1932; and Stevens, 1946) are conceptually naive and mathematically crude by comparison (with the notable exceptions of the little-known work in this area of Whitehead and Russell (1913) and Wiener (1914, 1915, 1921)). Philosophically, Hölder’s paper is of interest because it relates directly to modern work on the concept of quantity (e.g., Mundy, 1987; Michell, 1994; Swoyer, 1987). In addition, whatever his own view might have been, he shows that the identification made between ratios of magnitudes and real numbers suffers no formal impediment, connecting his paper to recent alternatives to representationalism in measurement theory (e.g., Michell, 1990, 1994; Niederée, 1987, 1992).

THE AXIOMS OF QUANTITY AND THE THEORY OF MEASUREMENT

O. Hölder

The axioms of arithmetic have hitherto been understood in two senses. In the first place, by “axioms of arithmetic” has been meant what I prefer to call “axioms of quantity” or “axioms of the theory of magnitudes”2 and these are the facts elucidated below. On the other hand, it is claimed that arithmetic, in the narrow sense of the arithmetic of the integers,3 is based on facts which are not able to be proved, i.e., axioms. So von Helmholtz calls the formula

\[ a + (b + 1) = (a + b) + 1, \]

on which H. Grassmann founds the addition of integers, an axiom of arithmetic.4 But this formula is only a description of the procedure of addition. It says that the number understood as the sum of \( a \) and \( b + 1 \) (the number following \( b \)) follows \( a + b \) in the number sequence. Consequently, this formula requires that \( a + 1, a + 2, a + 3, a + 4, \ldots, a + c \) be consecutive, and thus that the number \( a + c \) may be found by beginning at \( a + 1 \) (the successor of \( a \)) and taking each number consecutively until one has counted from \( 1 \) to \( c \). Hence, the above formula is better taken as defining a concept which is under our control and not as an axiom. Of course, one could take it to be an obvious but unprovable presupposition that this process of addition can always be carried out. Arithmetic contains other presuppositions of this sort, and they are based on our taking for granted that certain procedures, which we consider to follow explicit rules, can always be executed in a certain way and, in certain cases, continued indefinitely. In some cases it is obvious that a process will terminate, while in other cases this clearly must be proved. Each assumption which is understood as obvious in this way is closely related to the rules specifying the procedure involved, i.e., with the concept of procedure,4 and it seems to me that we cannot separate such presuppositions from the process of thinking in such a way that from a certain number of these presuppositions all lower and higher arithmetic can be derived without requiring any extra, similar presuppositions. Here we are dealing with a kind of experience which is not counted as sensory experience proper, since it can also be gained by the use of our thought processes alone. Basically, it is identical to the same (combinatorial) experience had when carrying out deductive proofs, which we occasionally substitute for the external experience involving the senses. Hence, it seems apt to label such products of thought as “purely logical” and so we can say that the arithmetic of integers can be developed using logic alone and that it does not require axioms.2 The same applies to the arithmetic of rational and irrational numbers when these are defined appropriately.

It is a different matter with geometry and mechanics, where certain axioms based upon sensory experience (or, as some want it, upon intuition5) are presupposed. As with geometry and mechanics, the theory of measurable magnitudes can be based upon a set of facts which I will call “axioms of magnitude” or “axioms of quantity.” The theory of measurable magnitudes applies in the same way to the comparison and addition of time, mass, line segments6 and area2. However, so as to prevent any misunderstanding from the outset, I note that the axioms of the theory of magnitudes shall not be presupposed for geometry in the form presented here, by applying them to line segments and area. On the contrary, the axioms of the theory of measurable magnitudes satisfied by line segments follow from purely geometrical axioms for points and segments on a line5 (see Part II of this paper).
The theory of measurable magnitudes was developed to a high level by Euclid. Recently, it has been treated in depth from a number of different points of view. Nevertheless, it seems that the theory has not been treated exhaustively; further, errors and obscurities have appeared in some of the more recent treatments. This is why I think that a reformulation of this important and fundamental theory will be profitable.

Part I: Magnitudes and Measure-Numbers

§1. THE AXIOMS

The axioms of quantity, i.e., the facts upon which the theory of measurable (absolute) quantities is based, are as follows: 4

I. Given any two magnitudes, \(a\) and \(b\), one and only one of the following is true: \(a\) is identical to \(b\) \((a = b)\), \(a\) is greater than \(b\) \((a > b)\), or \(b\) is greater than \(a\) \((b > a)\).

II. For every magnitude there exists one that is less.

III. For every ordered pair of (not necessarily distinct) magnitudes, \(a\) and \(b\), their sum, \(a + b\), is well-defined.

IV. \(a + b\) is greater than \(a\) and greater than \(b\).

V. If \(a < b\), then there exist \(x\) and \(y\) such that \(a + x = b\) and \(y + a = b\).

VI. It is always true that \((a + b) + c = a + (b + c)\).

VII. Whenever all magnitudes are divided into two classes such that each magnitude belongs to one and only one class, neither class is empty, and any magnitude in the first class is less than each magnitude in the second class, then there exists a magnitude \(\xi\) such that every \(\zeta < \xi\) is in the first class and every \(\zeta > \xi\) belongs to the second class. (Depending on the particular case, \(\xi\) may belong to either class.)

§2. SIMPLEST CONCLUSIONS FROM AXIOMS I TO VI

1. If \(a < a'\) and \(a' < a''\), then, according to V, there exist magnitudes \(x\) and \(x'\) such that \(a + x = a'\) and \(a' + x' = a''\). Therefore, \(a + x + x' = a''\) and, so, according to VI, \(a + (x + x') = a''\) and thus, from IV, \(a < a''\). Thus, from \(a < a'\) and \(a' < a''\) it follows that \(a < a''\).

2. Again, given \(a < a'\) and \(x\), such that \(a + x = a'\), then \(b + (a + x) = b + a'\) and, so, \((b + a) + x = b + a'\). Therefore, \(b + a < b + a'\). If \(y\) is here chosen such that \(y + a = a'\), then it follows that \((y + a) + b = a' + b\) and, so, \((y + a + b) = a' + b\), therefore, according to IV, \(a + b < a' + b\). Thus, from \(a < a'\) it follows that \(b + a < b + a'\) and that \(a + b < a' + b\) (where \(b\) is any magnitude).

Furthermore, if it is assumed that \(a < a'\) and \(b < b'\), then it follows that \(a + b < a' + b < a' + b'\); so we derive the theorem that smaller added to smaller yields smaller.

3. Let \(a < b\). Given \(x\), such that \(a + x = b\), and assuming \(x' < x\), which is allowed by II, it follows (by §2, No. 2) that \(a + x' < a + x\), i.e., \(a < b\). On the other hand, by IV, \(a + x' > a\). Hence, since \(a < b\), there exists at least one magnitude which is \(> a\) and \(< b\); i.e., there is at least one magnitude between \(a\) and \(b\).

4. Given any magnitude, \(a\), there is one that is greater, since, e.g., \(a + a > a\).

5. The magnitude, \(x\), postulated in axiom V, is uniquely determined (uniqueness of the one kind of subtraction). 5 For if both \(a + x = b\) and \(a + x' = b\), then \(a + x = a + x'\). However, if \(x > x'\) or \(x < x'\), then the last equation contradicts No. 2 of this paragraph. According to axiom I the only possibility is \(x = x'\).

That the magnitude, \(y\), postulated in V, is uniquely determined may be proven in a similar manner (uniqueness of the other kind of subtraction).

§3. MULTIPLICATION

1. To unambiguously specify the formation of multiples of a magnitude I will set

\[2a = a + a,\quad 3a = (a + a) + a,\quad 4a = ((a + a) + a) + a,\quad \text{etc.}\]

so that in general,

\[na = (n - 1) + a.\]

Now, it is well known that the associative law of addition, given for three magnitudes in axiom VI, holds for any number of magnitudes. If one has \(m + n\) magnitudes, all equal to \(a\), then it can be seen that the equation

\[ma + na = (m/n) a\]

is true for every magnitude, \(a\), and for any two (positive) integers \(m\) and \(n\).

It follows, through repeated application of equation (1), that

\[ma + ma + ma + \cdots = (m + m + m + \cdots) a,\]

where the sums on the right and on the left have \(m'\) summands. Consequently,

\[m'(ma) = (m'm) a,\]

for any two integers, \(m\) and \(m'\).
2. It follows from (1), with the help of axiom IV, that 

$$(m + n) a > ma$$

and so, it can be seen that \(m^2 a > \), or \(< ma\) only if \(m^2 <\), =, or \(< m\), for the integers, \(m\) and \(m\). In

particular, it follows from \(m^2 a = ma\) that the integers \(m\) and \(m\) are equal.

By repeated application of \(\S\), No. 2, it can also be seen that \(ma < \), =, or \(> mb\) depends upon whether \(a <\), =, or \(> b\); in particular, one can conclude from \(ma = mb\) that the magnitudes \(a\) and \(b\) are equal whatever integer \(m\) might be.

3. Given any magnitude, \(a\), and integer, \(n\), one can always find a magnitude, \(b\), such that \(nb < a\). First, according to \(\II\), one can find \(a' < a\) and with it \(a''\) such that \(a' + a'' = a\).

If \(a\) is then selected such that it is less than both \(a'\) and \(a''\), i.e., less than the smaller of the magnitudes, \(a'\) and \(a''\), then according to \(\SS\), No. 2, the sum \(a_1 + a_2 < a' + a''\), i.e., \(2a_1 < a\). Similarly, \(a_2\) can be determined such that \(2a_2 < a_1\), \(a_3\) such that \(2a_3 < a_2\), etc. If the integer \(v\) is now chosen such that \(2^v < n\) and one sets \(a = b\), then \(nb < a\).

Q.E.D.

$\S 4$. THE ARCHIMEDEAN AXIOM

Let \(a\) and \(b\) be two magnitudes, with \(a < b\). Our aim is to prove that there exists an integer, \(n\), such that \(na > b\). First assume the opposite, that for every integer, \(n\), the magnitude \(na \leq b\). One can conclude every magnitude smaller than some multiple of \(a\) to a first class and all other magnitudes to a second class. Since, for example, \(a\) belongs to the first and \(b\) to the second class, neither is empty. If \(c\) is a magnitude in the second class, then \(na < c\) for every integer, \(n\). If \(c\) belongs to the first class, then there exists an integer, \(n_1\), such that \(c_1 < n_1 a\) and since, at the same time, for this \(n_1\), the inequality, \(n_1 a < c\), obtains, it is the case that \(c_1 < c\).

So every magnitude of the first class is less than each magnitude of the second class. It then follows from VII that there exists a magnitude \(z\) such that every \(z'\) less than \(z\) belongs to the first class, and every \(z''\) greater than \(z\) belongs to the second class.

A multiple of \(a\) cannot be greater than or equal to \(z\). For

if \(n_1 a > z\) then there would be \((\SS, No. 3)\) a magnitude between \(n_1 a\) and \(z\), which, since it would be \(<n_1 a\), would have to belong to the first class, and since it would be \(> z\), it would also have to belong to the second class, which would be a contradiction. But if \(n_1 a = z\), then (equation (1)) the next multiple, \((n_1 + 1) a = n_1 a + a\), according to IV, would be greater than \(n_1 a\), i.e., greater than \(z\), which contradicts what has just been found. Thus, for every integer \(n\), \(na < z\).

Now consider \(a' < a\) (axiom II); from the above we have \(1 - a\) or \(a\) is less than \(z\) and, so, \(a' < z\) as well (\(\SS\), No. 1). One can now determine \(c\) such that \(z' + a'' = z\) (axiom V), which implies that \(z'' < z\) (axiom IV). Since \(z''\) must belong to the first class, one can find an integer, \(n'\), such that \(n'a > z'\).

From this inequality and from \(a > a'\), it follows that \(n'a + a > z' + a'\) (see \(\SS\), No. 2), i.e., \((n' + 1) a > z\). This contradicts what has already been proved.

Therefore the initial assumption is impossible; i.e., there exists an integer, \(n\), such that \(na > b\). This fact is often specified as a particular axiom and referred to as the Archimedean axiom.

$\S 5$. THE COMMUTATIVE LAW OF ADDITION

$\S 1$ left unresolved whether or not the commutative law of addition obtained. It will now be proved that the equation, \(a + b = b + a\), follows necessarily from axioms I to VII.

Let \(c\) be chosen, the only constraint being that it \(< a\) and \(< b\). The magnitudes, \(2, 3c, 4c, \ldots\) are, according to the Archimedean axiom, not all \(< a\). Let the first magnitude of this sequence which is \(> a\) be \(\mu\). Therefore,

$$\begin{align*}
(\mu - 1) a & \leq a, \quad \text{and} \\
\mu c & > a.
\end{align*}$$

Similarly, an integer, \(v\), must exist such that

$$\begin{align*}
(v - 1) a & \leq b, \quad \text{and} \\
v c & > b.
\end{align*}$$

From (3) and (5) it follows, according to \(\SS\), No. 2, that

$$(\mu - 1) c + (v - 1) c \leq a + b.$$  

Thus, because of (1),

$$(\mu + v - 2) c \leq a + b.$$  

Just as this relation was deduced from (3) and (5), so it follows from (6) and (4) (note the order of the addition) that

$$(v + \mu) c > b + a.$$  

Since the commutative law of addition holds for numbers, one then has

$$(\mu + v) c > b + a.$$  

From (7), it also follows that \((\mu + v - 2) c + 2c \leq (a + b) + 2c\), which, considering (1), results in the relation

$$(\mu + v) c \leq (a + b) + 2c.$$  

According to \(\SS\), No. 1 it follows from (8) and (9) that

$$b + a < (a + b) + 2c.$$  

(10)
Hence, one can see that it cannot be the case that $b + a > a + b$; for if this were the case, there would be an $x$ such that

$$(a + b) + x = b + a. \quad (11)$$

Apart from being $<a$ and $<b$, the magnitude $c$ was arbitrarily chosen; hence, it could also be chosen (§3, No. 3) such that $2c < x$. In this case, it would follow that $(a + b) + 2c < (a + b) + x$, i.e., according to (11), less than $b + a$, which contradicts (10).

Since, throughout the preceding considerations, the roles of the magnitudes, $a$ and $b$, can be interchanged, it also follows that $a + b > b + a$ cannot be true. Therefore (from axiom I), it must be true that $a + b = b + a$.

### §6. DEDUCTIONS FROM THE COMMUTATIVE LAW OF ADDITION

1. From the commutative law of addition, together with the associative law, it follows that

$$(a + b) + (a + b) + (a + b) + \cdots = (a + a + a + \cdots) + (b + b + b + \cdots);$$

i.e., that the equation

$$m(a + b) = ma + mb \quad (12)$$

obtains for any magnitudes, $a$ and $b$, and for all integers, $m$.

2. Let $\mu$ and $\nu$ be two (positive) integers, $a$ and $b$, two magnitudes and let $\nu a < \mu b$. We aim to prove that there exists a magnitude, $\alpha > a$, such that $\nu a < \mu b$; a magnitude, $b' < b$, such that $\nu a < \mu b'$; and two integers, $\mu'$ and $\nu'$, so that as well as the relation, $\nu' a < \nu' b'$, the inequality, $\nu' a < \mu' b'$, obtains.

To prove the first part of this assertion, let

$$va + x = \mu b \quad (13)$$

and select $x'$, such that (§3, No. 3) $\nu x' < x$. Then, from (12),

$$\nu(a + x') = va + \nu x'$,

which is $< \nu a + x$, i.e., $< \mu b$; so, for $a' = a + x$, this part of the assertion is correct.

To prove the second part, set $x$ as before, $x''$, such that both $\mu x'' < x$ and $x'' < b$, and let $b'$ be defined by the equation

$$b' + x'' = b, \quad (14)$$

whereby $b' < b$. From (13) and (14), according to (12), it follows that

$$va + x = \mu b' + \mu x''.$$
The applicability of numbers to magnitudes can be established only by means of the proofs just mentioned (which will be carried out below) if the assumption is made that the magnitudes discussed satisfy axioms I to VII; at the same time, the Euclidean theory of proportions is brought into close connection with the concept of measurement and the modern arithmetical theory of irrational numbers.

§ 9. RATIONAL AND IRRATIONAL NUMBERS

As already mentioned, the purely arithmetical concept of rational and irrational numbers will be employed below. Concerning rational numbers, it should be noted that a fraction, \( m/n \), is to be regarded solely as a form specified by the positive integer, \( m \), in the numerator and the positive integer, \( n \), in the denominator. It is stipulated that \( m/n = m'/n' \) whenever \( mn' = m'n \) and \( m/n > m'/n' \) whenever \( mn' > m'n' \). From this definition it can be shown that two fractions, equal to a third, must be equal to each other and the convention is adopted that all fractions equal to a specific fraction represent the same object, which we call a rational number. Given appropriate definitions of sums and products, the known arithmetic laws for these rational numbers can be proved.

Through the investigations of Weierstrass, Dedekind, and G. Cantor, the irrational numerical magnitudes (which, for the sake of simplicity, we will also refer to as “numbers”) have, also, been placed on a purely arithmetical foundation. In the following, Dedekind’s theory is adopted because it is the most convenient for our purpose. According to this theory, an irrational number is defined by a “cut”; i.e., it is defined by specifying all rational numbers greater than it. Since this theory has been adequately established and is sufficiently well-known, I will only informally mention some of its definitions and theorems, without supplying the proofs, which can be given purely arithmetically.

1. A partition of all (positive, non-zero) rational numbers into two classes is called a cut when each rational number is assigned to one and only one class, when neither class is empty, when each rational number in the first class is less than any in the second class, and when the first class contains no greatest rational number.

2. The rational numbers in the first class shall be called the lower numbers of the cut and those of the second class shall be called upper numbers of the cut. If the upper numbers include a least element, it shall be called an improper upper number. An upper number is called a proper upper number if the cut under consideration contains an upper number less than it.

3. Given two cuts, where \( a \) and \( b \) designate any lower numbers of the first and second cuts, respectively, then all
numbers \( a + b \) are identical to all numbers of a third cut, which is called the sum of the first and second cuts.

4. If the proper upper numbers of the first and second cuts mentioned in No. 3 are designated by \( a' \) and \( b' \), respectively, then all numbers of the form \( a' + b' \) are identical to all proper upper numbers of the third cut.

5. Employing terms as introduced in Nos. 3 and 4, the numbers, \( ab \), are identical to the lower numbers of a fourth cut, known as the product of the first and second cuts. The proper upper numbers of the fourth cut are the numbers, \( a'b' \).

6. Given two non-identical cuts, one and only one of them contains a lower number which is an upper number of the other; this cut is said to be greater than the other.

All known arithmetical laws apply to cuts: the associative and commutative laws of addition and multiplication, the distributive law of the form \( ab + c = ab + ac \); the theorem that larger added to larger results in larger, etc. Also, the laws for inverse operations hold (e.g., for division, which is easily defined) and all of these laws can be proved purely arithmetically.

Given two cuts, where the least upper number of the first is \( r \) and the least upper number of the second is \( r' \), the least upper number of the sum of these cuts is the rational number \( r + r' \) and the least upper number of the product of these cuts is \( rr' \). Calculations with cuts of this special kind parallel calculations with rational numbers and these cuts can be considered as representing rational numbers and even identified with them.

A cut without a least upper number is taken to represent an irrational number and might simply be called an irrational number.

\[\text{§10. The Number (Measure-Number) Corresponding to a Ratio of Magnitude}\]

Let \( a \) and \( b \) be two magnitudes of a kind to which axioms I to VII apply. According to §3, multiples of magnitudes have a well-defined meaning. If for positive integers, \( \mu \) and \( v \), the inequality, \( va > \mu b \), obtains, then \( \mu / v \) shall be called a lower fraction relating to the ratio, \( a : b \). In the other case, when \( va \leq \mu b \), \( \mu / v \) shall be called an upper fraction relating to the ratio, \( a : b \).

If \( \mu / v \) is a lower and \( \mu' / v' \) an upper fraction relating to the given ratio, then we have \( va > \mu b \) and \( \mu b \geq v'a \), from which the relations \( (\mu' / v') a \geq (\mu / v) b \) and \( (\mu / v) b \geq (\mu' / v') a \) follow by §3, No. 2 and equation (2). From these relations it follows that \( (\mu' / v') a \geq (\mu / v) a \) and so (by §3, No. 2) \( \mu' / v' \geq \mu / v \) as well; i.e., the fraction \( \mu / v \) is less than \( \mu' / v' \).

Therefore, every lower fraction is less than each upper fraction relating to the same ratio. In particular, two equal fractions must be either both lower or both upper fractions, which is why every rational number is either an upper or lower fraction relating to the ratio \( a : b \), regardless of how it is expressed as a fraction. Also, every lower rational number is less than each upper one. Since (according to the Archimedean condition) there exists an integer, \( v \), such that \( va > \mu b \), as well as a number, \( \mu \), such that \( 1a < \mu b \), there must be both upper and lower numbers relating to every ratio.

For any fraction, \( \mu / v \), such that \( va > \mu b \), one can find (according to §6, No. 2) a larger fraction, \( \mu' / v' \), for which it is also true that \( v'a > \mu' b \); i.e., for every lower rational number there is a larger one that is also lower. Consequently, there is no greatest lower number. It is now obvious that the ratio, \( a : b \), has become associated with the kind of division of all rational numbers which we call a cut (§9, No. 1). Therefore, this proposition follows:

For each ratio of magnitudes, \( a : b \), i.e., for each two magnitudes taken in a specific order, there exists a well-defined cut, i.e., a definite number24 in the general sense of the word. This number shall be denoted \([a : b]\).

One can also call the number, \([a : b]\), the measure-number obtained when magnitude \( a \) is measured by magnitude \( b \), in which case \( b \) is called the unit. From §3, No. 2 it then follows:

The cut belonging to the ratio of \( a : a \) represents the number 1 (see the end of §9), i.e., \([a : a] = 1\).

\[\text{§11. Commensurable Magnitudes}\]

1. Two magnitudes, \( a \) and \( b \), are called commensurable24 when there are two integers, \( \mu \) and \( v \), such that \( va = \mu b \). In this case \( \mu / v \) is an upper fraction relative to the ratio \( a : b \) (see §10). If \( \mu' / v' \) is also an upper fraction, then \( v'a < \mu' b \) and it follows (§3, No. 2 and equation (2)) that \( (v' \mu) b = (v' \mu) a = (v' \mu) a \leq (\mu' v) b \). Thus, \( \mu \leq v' \mu \); i.e., the fraction, \( \mu / v \), is less than or equal to the fraction, \( \mu' / v' \). From this it follows that \( \mu / v \) is the least upper rational number relating to the ratio, \( a : b \). Hence, this theorem results:

If \( a \) and \( b \) are commensurable and \( va = \mu b \), then the cut corresponding to the ratio, \( a : b \), has a least upper number, namely the rational number \( \mu / v \). Consequently (§10, and the end of §9), one may set \([a : b] = \mu / v\).

2. The opposite must still be proved. Suppose that a least upper number \( \mu / v \) belongs to the cut corresponding to the ratio, \( a : b \). According to the definition of upper numbers, we have \( va \leq \mu b \). We must show that here \( va = \mu b \). If \( va < \mu b \), then one could, according to §6, No. 2, find a fraction \( \mu' / v' < \mu / v \), such that \( v'a < \mu' b \). Then \( \mu' / v' \) would be both an upper fraction and less than the least upper fraction. Hence we have the theorem:
If the cut corresponding to the ratio, \( a:b \), has a least upper number, \( \mu/\nu \), then \( va = \mu b \).

3. It also follows from No. 1 of this section that if the fraction, \( \mu/\nu \), is an upper number but not the least upper number for any commensurable or incommensurable ratio, \( a:b \), then it cannot be true that \( va = \mu b \). Thus, for every proper upper (see §10 and §9, No. 2) fraction, \( \mu/\nu \), of any ratio, \( a:b \), the inequality, \( va < \mu b \), obtains.

§12. THE MEASURE-NUMBER OF A SUM OF MAGNITUDES

1. Let the three magnitudes, \( a, a', \) and \( b \), be given. The ratio of \( a:b \) defines a first cut, the ratio of \( a':b \) a second, and these two jointly determine a third, which is their sum.

Each lower number of the third cut can (§9, No. 3) be represented by

\[
\mu/\nu + \mu'/\nu' = (\nu'\mu + \nu\mu')/\nu
\]

(17)

where \( \mu/\nu \) and \( \mu'/\nu' \) are lower fractions of the first and second cuts, respectively.

According to the definition of lower fractions (§10), \( va < \mu b \) and \( \nu' a' > \mu' b \), from which it follows (§3, Nos. 2 and 1) that \( (\nu'\nu) a > (\nu'\mu) b \) and \( (\nu'\nu) a > (\nu'\mu') b \). From §2, No. 2 and from equations (12) and (1), it follows, as well, that

\[
(v'\nu)(a + a') > (\nu'\mu + \nu'\mu') b
\]

This inequality asserts nothing more than that fraction (17) is a lower fraction of the ratio, \( (a + a'):b \).

Every proper upper number of the third cut can (§9, No. 4) be expressed in the form

\[
p/\sigma + p'/\sigma' = (\sigma' p + \sigma p')/\sigma \sigma',
\]

(18)

where \( p/\sigma \) and \( p'/\sigma' \) are proper upper fractions of the first and second cuts. One can now (§11, No. 3) say that \( sa < pb \) and \( \sigma a' < \rho b' \) and deduce from this that \( (\sigma'\sigma) a < (\sigma'\rho) b \) and \( (\sigma'\sigma') a < (\sigma'\rho') b \), from which it follows again that

\[
\sigma'(a + a') < (\sigma'\rho + \sigma\rho') b.
\]

This inequality implies (from §10; §11, No. 2; and §9, No. 2) that (18) is a proper upper fraction relating to the ratio, \( (a + a'):b \).

Taking the cut corresponding to the ratio, \( (a + a'):b \), as the fourth cut, then what has been shown so far is that any lower number, \( \alpha \), of the third cut is also a lower number of the fourth cut, and every proper upper number, \( \beta \), of the third cut is also a proper upper number of the fourth cut. If the third cut has an improper upper number, \( \xi \), it has as yet not been shown to which class formed by the fourth cut \( \zeta \) belongs. Now, \( \zeta \) is a rational number falling between the numbers, \( \alpha \), and the numbers, \( \beta \), and \( \xi \), together with \( \alpha \) and \( \beta \), comprises the totality of all rational numbers. Thus, if \( \xi \) were a lower number of the fourth cut, then the class of these lower numbers would obviously consist only of the numbers, \( \alpha \), and \( \xi \); therefore this class would contain a greatest number, namely \( \xi \), which contradicts §9, No. 1. Clearly, the third cut must be identical with the fourth cut. Consequently, this proposition holds:

The cut corresponding to the ratio, \( (a + a'):b \), is the sum of the two cuts belonging to the ratios, \( a:b \) and \( a':b \).

Put in other words this means:

The measure-number of the sum of magnitudes, \( a + a' \), is the arithmetical sum of the measure-numbers of \( a \) and \( a' \), provided that \( a, a' \) and \( a + a' \) are all measured relative to the same unit.

In terms of the notation introduced in §10 this result can also be expressed by the formula

\[
[(a + a'):b] = [a:b] + [a':b].
\]

(19)

2. If \( a_1 > a \), then \( a_1 \) may be put in the form, \( a + a' \), and so, by formula (19), it is evident that \( [a_1:b] = [(a + a'):b] > [a:b] \). Finally, the following proposition results:

\[
[a_1:b], = 2, or < [a_2:b]
\]

depending upon whether \( a_1 > s, =, or < a_2 \). More specifically, from the equation \( [a_1:b] = [a_2:b] \) one can always conclude that \( a_1 = a_2 \). Moreover, since \( [b:b] = 1 \) (see the end of §10), \( [a:b] >, =, or < 1 \) depending upon whether \( a >, =, or < b \).

§13. CHANGE OF UNIT

From any three magnitudes, \( a, b, \) and \( c \), the cuts designated by \( [a:b] \) and \( [b:c] \) may be formed (§10); call \( [a:b] \) the first cut and \( [b:c] \) the second. Let the third cut be the product of these two. Now every lower number (§9, No. 5) of the third cut is of the form

\[
\mu k/\nu \lambda,
\]

(20)

where \( \mu/\nu \) and \( k/\lambda \) are lower fractions of the ratios, \( a:b \) and \( b:c \). Thus, given the definition of lower fractions (§10), the inequalities, \( va > \mu b \) and \( \nu' a' > \mu' b' \) and \( \sigma a' < \rho b' \) hold. From this it follows that (§3, No. 2 and equation (2)) \( (\nu'\nu) a > (\nu'\mu) b \) and \( (\nu'\nu) a > (\nu'\mu') b \), whence, \( (\nu'\nu) a > (\nu'\mu) c \) as well; i.e. (20) is a lower fraction of the ratio, \( a : c \).

In the same way, it can be proved that every proper upper number (§9, No. 2; §10; §11, No. 3) of the third cut is
simultaneously a proper upper number of the ratio, \(a : c\), and so it can be shown that the third cut is identical to the one corresponding to the ratio, \(a : c\), just as the analogous result was established in the preceding paragraph. This result is expressed by the formula

\[
[a : b] \cdot [b : c] = [a : c],
\]  
(21)

which states that one obtains the measure-number of \(a\) relative to \(c\) as a unit when the measure-number of \(a\) relative to unit \(b\) is multiplied with the measure-number of \(b\) relative to unit \(c\).

Putting formula (21) in this form,

\[
[a : b] = [a : c]/[b : c],
\]  
(22)

allows the calculation of the measure-number of any magnitude \(a\) relative to any magnitude \(b\) chosen as unit, provided that the measure-numbers of both magnitudes relative to any other unit \(c\) are known.

Furthermore, if one sets \(c = a\), then it follows (see the end of §10) that \([a : b]\) and \([b : a]\) are reciprocal numerical values.

**§14. CONTEMPORARY ACCOUNT OF THE THEORY OF THE PROPORTIONS OF MAGNITUDES**

The results of the last section, especially equations (19) and (22), reduce the Euclidean theory of proportions to laws of arithmetic and, thereby, give the contemporary account greater transparency.

As an example, take Theorem XII of Book 5 of the Elements. This theorem implies that if magnitude \(a\) relates to \(b\) as the ratio of \(c\) to \(d\) and these, as \(e\) to \(f\), then \(a + c + e\) relates to \(b + d + f\) as the ratio of \(a\) to \(b\). It is assumed that these numbers are equal,

\[
[a : b], \ [c : d], \ [e : f].
\]  
(23)

According to (22), these numbers are equal to

\[
[a : g]/[b : g], \ [c : g]/[d : g], \ [e : g]/[f : g]
\]  
(24)

respectively, where \(g\) is an arbitrary magnitude. But we can prove in a purely arithmetic way that three equal quotients (24) are numerically equal to

\[
([a : g] + [e : g])/(b + [d : g]) + [f : g]).
\]  
(25)

According to (19) this is equal to

\[
[(a + c + e) : g]/[(b + d + f) : g],
\]

i.e., (see (22)) equal to

\[
[(a + c + e)]/[(b + d + f)],
\]  
(26)

and, so, the numbers (24) and (23), which are identical with (25), are equal to (26); i.e.,

\[
[(a + c + e)]/[(b + d + f)] = [a : b]
\]  
Q.E.D.

The proof given here rests on the fact that, with the help of (22), all measure-numbers are expressed relative to exactly the same unit and that equations (19) and (22) can then be used. If the Euclidean proof is compared with this, then the difference rests on the fact that Euclid must fall back on his definition of proportion,25 both here and with every other proof in his theory of proportion; this would be the same as us having to revert to the definition of a cut in each instance.

**§15. THE EXISTENCE OF A MAGNITUDE OF A PRESCRIBED MEASURE-NUMBER**

Relative to a given magnitude, \(b\), and a given cut, \(\kappa\), all magnitudes belong to two classes as follows. In the first class belong those magnitudes, \(x\), which, together with a suitably chosen lower fraction (§9, No. 2), \(\mu/\nu\), of the cut, \(\kappa\), satisfy the inequality

\[
\nu x < \mu b.
\]  
(27)

Every other magnitude belongs in the second class, so that every magnitude, \(x'\), of this class, together with every lower fraction, \(\mu'/\nu'\), of the cut, \(\kappa\), satisfies the relation,

\[
\nu x' \geq \mu b.
\]  
(28)

If \(x\) and \(x'\) are two specific magnitudes of the first and second classes, respectively, then inequality (27) holds for the specific magnitude, \(x\), and for two integers, \(\mu\) and \(\nu\), and relation (28) holds for these numbers, \(\mu\) and \(\nu\), amongst others, and for the specific magnitude, \(x'\). One can therefore deduce from (27) and (28) that \(\nu x < \nu x'\) and consequently, via §3, No. 2, that \(x < x'\), as well. Thus, every magnitude of the first class is less than each of the second.

In order to see that there are magnitudes in both classes, we first choose a lower fraction, \(\mu/\nu\), of cut, \(\kappa\). According to §3, No. 3, one can find a magnitude, \(x\), such that \(\nu x < \mu b\), in which case, \(x\) belongs to the first class. If we now select an upper fraction, \(\mu'/\nu'\), of this cut and a magnitude, \(x'\), such that \(x' < \mu'b\) (e.g., §2, No. 4), then we also find that

\[
\nu x' > \mu b.
\]  
(29)
Now, if \( \mu / \nu \) is any lower fraction of the cut, \( \kappa \), then (§9, No. 1 and No. 2) \( \mu / \nu < \mu'/ \nu' \); i.e., \( \mu / \nu < \mu'/ \nu' \). It therefore follows from (29) that \( (\mu/ \nu)' > (\mu'/ \nu') b \) and, thus, a fortiori (see §3, No. 2) that \( (\mu'/ \nu') x' > (\mu/ \nu) b \), which means (because of equation (2)) that \( \mu'/ (\nu' x') > \mu/ \nu b \). Therefore (§3, No. 2), it must be true that \( x' > \mu b \) for any lower fraction, \( \mu/ \nu \), and so \( x' \) belongs to the second class.

By dint of axiom VII, one can now deduce the existence of a magnitude, \( \xi \), such that every \( \xi' \) which is \( < \xi \) belongs to the first and each \( \xi'' \) which is \( > \xi \) belongs to the second class. It remains to be shown that \( \kappa \), the cut given at the beginning, corresponds to the ratio, \( \xi : b \).

First we let \( \mu / \nu \) be a lower fraction of cut, \( \kappa \); it must then be proved that \( \mu / \nu \) is also a lower fraction relating to the ratio, \( \xi : b \), i.e. (§10), that \( \nu \xi' > \mu b \). In order to do this, let us assume the opposite, viz., that

\[
\nu \xi' \leq \mu b. \tag{30}
\]

According to the properties of lower fractions (§9, Nos. 1 and 2), there must be in cut \( \kappa \) a fraction, \( \mu'/ \nu' \), which again is a lower fraction, such that \( \mu'/ \nu' > \mu'/ \nu' \); i.e., \( \nu / \nu' > \nu'/ \nu' \). It now follows from (30) that \( (\nu/ \nu') \xi' < (\nu/ \nu') b \) and, since (§3, No. 2), \( (\nu/ \nu') x' < (\nu/ \nu') b \), it must also be the case that \( \nu / \nu' \xi' < (\nu/ \nu') b \); i.e. equation (2)), \( \nu / \nu' \xi' > (\mu/ \nu) b \) holds. It follows, therefore, from (§3, No. 2), that \( \nu / \nu' \xi' < \mu b \). According to §6, No. 2, one can now find a magnitude, \( \xi'' > \xi \), such that the last inequality remains true when \( \xi'' \) is substituted for \( \xi \). Accordingly, one obtains \( \nu / \nu' \xi'' < \mu b \). Since \( \mu'/ \nu' \) was a lower fraction of cut \( \kappa \), then according to the definition given at the beginning of this section, \( \xi'' \) would belong to the first class. Since by assumption \( \xi'' > \xi \), this result contradicts the properties of magnitude \( \xi \).

Now, let \( \rho / \sigma \) be any proper or improper upper fraction of cut \( \kappa \). It must be proved that \( \rho / \sigma < \rho / \sigma \); i.e., \( \sigma \xi < \rho b \). Suppose that \( \sigma \xi > \rho b \), then \( \xi'' < \xi \) could be chosen such that simultaneously (§6, No. 2) \( \sigma \xi'' > \rho h \). Now, if \( \mu / \nu \) is any lower fraction of cut \( \kappa \), then \( \mu / \nu < \rho / \sigma \), i.e., \( \sigma \mu < \nu \rho \), and it follows from this that \( (\nu/ \nu') \xi'' > (\nu/ \nu') b \) which therefore from this that \( \nu / \nu' \xi'' > \mu b \). From this inequality, which, for the same \( \xi'' \), would have to be valid for all lower fractions, \( \mu / \nu \), of the given cut, \( \kappa \), one obtains (see (28)) that magnitude \( \xi'' \) should belong to the second of the classes defined earlier. But since \( \xi'' < \xi \) was chosen, this result contradicts the properties of magnitude \( \xi \).

Hence, this proposition has been proved:

There exists exactly one magnitude, \( \xi \), whose ratio to an arbitrarily given magnitude, \( b \), is specified by some arbitrarily chosen cut, \( \kappa \).

This may also be put as follows:

There exists exactly one magnitude, \( \xi \), which relative to an arbitrarily given unit, \( b \), has some arbitrarily chosen number, \( \kappa \), as its measure-number.

The magnitude, \( \xi \), is uniquely determined by the unit, \( b \), and the measure-number, \( \kappa \), so if two magnitudes, \( \xi \) and \( \eta \), satisfy this condition, then employing the notation of §10 one has the relation, \( [\xi : b] = \kappa = [\eta : b] \), from which it follows by §12, No. 2, that \( \xi = \eta \).

If the unit is fixed, then there is a measure-number for each given magnitude and there is a magnitude for each given measure-number. 29

§16. MULTIPLICATION OF MAGNITUDES

Nothing else has been assumed about the magnitudes considered so far, other than axioms I to VII, in particular, that these magnitudes can be compared and added. The issue of the multiplication\(^{30} \) of magnitudes has not been considered. But an operation can now be defined which may be called multiplication.

Throughout the following discussion, a fixed magnitude, \( b \), called the unit, will be adhered to. If \( a \) and \( a' \) are any two magnitudes, then I define \( a \times a' \) as that magnitude, \( x \), for which the numerical equation

\[
[x : b] = [a : b][a' : b] \tag{31}
\]

holds. According to §15 (conclusion), and by dint of this stipulation, the product \( a \times a' \) of magnitudes, \( a \) and \( a' \), is uniquely determined; but this determination would change if \( b \) were changed.

Since the commutative law of multiplication holds for numbers \( [a : b], [a' : b] \) it follows directly from the defining equation (31) that \( aa' = a'a \) holds for magnitudes. Furthermore, if we wish to construct \( a(a' a) \) it will first define \( x \) in accordance with equation (31) and then define \( x' \) by the equation \( [x' : b] = [x : b][a' : b] \). Thus, \( x' \) can be found directly by the equation

\[
[x' : b] = [a : b][a' : b][a'' : b].
\]

So, it is also evident that

\[
(a(a')) \_ = a(a' a'),
\]

i.e., the associative law holds for the multiplication of magnitudes so defined.

The magnitude, \( (a(a' + a'')) \_ = y \), is determined by the equation

\[
[y : b] = [a : b][(a' + a'') : b].
\]

According to equation (19), the right side equals \( [a : b] [(a' + a'') : b] \); i.e., since the usual laws of calculation hold for numbers, equals \( [a : b][a' : b] + [a : b][a'' : b] \). The terms of the sum, according to the defining equation (31), denote the products, \( [(aa') : b] \) and \( [(aa'') : b] \)
respectively, and together give, according to equation (19),
\[(aa' + aa') : b\]. Thus,
\[y : b = [(aa' + aa') : b],\]
and therefore, according to §12, No. 2, also \(y = aa' + aa'\); i.e., for the three arbitrarily given magnitudes,
\[a(a' + a') = aa' + aa'. \quad (32)\]
Because of the commutative law of multiplication, it further results that
\[(a' + a) : a = a'a + a'\]
and, so, both distributive laws also hold.
Since the unit \(b\) was arbitrary, the multiplication shown here can be further determined in an infinite number of ways. If one bases the definition of multiplication upon unit magnitudes, \(a\), \(b\), and \(c\), then the defining equation (31) can be expressed in the form (see equation (21))
\[(x : c) = [(c : b) : (a : c)](a' : c).\]
and, consequently, also in the form
\[(x : c) = (c : b) : [(a : c)](a' : c).\]
Here, \(c\) is arbitrary and since \(b\) was also arbitrary, \(c : b = \kappa\) is an arbitrary number. Therefore, one can employ any arbitrarily chosen magnitude, \(c\), and arbitrarily chosen number, \(\kappa\), to define for all pairs of magnitudes, \(a\) and \(a'\), one of the multiplications under consideration by the equation
\[(aa') : c = \kappa \cdot (a : c) : (a' : c). \quad (34)\]

§17. THE DEFINED MULTIPLICATION IS UNIQUE

I now assume that from somewhere an operation is given which enables us to derive from two magnitudes, \(a\) and \(a'\), taken in a specific order, a uniquely determined third magnitude, \(a \cdot a'\). This operation shall also be called multiplication, and for this multiplication, together with the earlier defined addition, the two distributive laws (32) and (33) are taken to hold, so that
\[a \cdot (a + a') = (a \cdot a') + (a \cdot a') \quad \text{and} \quad (a' + a) \cdot a = (a' \cdot a) + (a' \cdot a) . \quad (35)\]
It can be shown that this new multiplication must also satisfy the commutative and associative laws and that it is nothing other than the multiplication of the previous section based upon a suitable unit.

1. In the first place conclusions will be deduced from the first distributive law (35). Let \(a\) be the lesser of two distinct magnitudes; the greater, \(a'\), can then be expressed in the form, \(a + a'\) (axiom V). It then follows from (35) and IV that
\[b \cdot a' = (b \cdot a) + (b \cdot a') > b \cdot a,\]
where \(b\) is any other magnitude. The new product is therefore enlarged by enlarging the second factor.
If one now forms the sum consisting of \(v\) parts,
\[(b \cdot a) + (b \cdot a) + (b \cdot a) + \cdots ,\]
by adding the first term to the second, and to the sum thus obtained, adding the third term, etc., one obtains through the constant use of formula (35), \(b \cdot va\); consequently,
\[v(b \cdot a) = b \cdot va. \quad (37)\]
Let \(c\) be any magnitude and \(\mu / v\) a lower fraction for \(a : c\) (see §10), such that \(va > \mu c\). According to what has just been found above, it is now also true that \(b \cdot va > b \cdot \mu c\); thus, according to (37), \(v(b \cdot a) > \mu (b \cdot c)\). But this relationship means that \(\mu / v\) is a lower fraction for \((b : a) : (b : c)\).
One can also prove that any upper fraction for \(a : c\) is also an upper fraction for \((b : a) : (b : c)\). Therefore, the cut defined by \(a : c\) is the same as the one defined by \((b : a) : (b : c)\), which may be expressed by equation
\[(b : a) : (b : c) = [a : c]. \quad (38)\]

2. In a completely analogous fashion, deductions may be drawn from the second distributive law (36). This yields, in the first place, that the present product is enlarged by enlarging the first factor, and then that equation
\[(b : c) \cdot (d : c) = [b : d] \quad (39)\]
holds for three arbitrary magnitudes, \(b\), \(c\), and \(d\).

3. Earlier we postulated the validity of both distributive laws for the new multiplication and, so, equations (38) and (39) jointly give us
\[(b : a) : (b : c) : (d : c) = [a : c] : [b : d]\]
or, considering equation (21) of §13,
\[[b : a] : (d : c) = [a : c] : [b : d].\]
If one then sets \(d = c\), it follows that
\[[b : a] : (c : c) = [a : c] : [b : c].\]
If one multiplies both sides by the number, \([c : c : c]\), and at the same time reappears equation (21), one gets

\[
[(b \cdot a) : c] = \kappa [(b : c) (a : c)],
\]

(40)

where

\[
\kappa = [(c : c : c)]
\]

which, being a number dependent only upon magnitude \(c\), is constant relative to magnitudes \(a\) and \(b\). Thus, the measure-number of \(b \cdot a\) is equal to the product of the variable measure-numbers of \(b\) and \(a\) times a constant given through the unit.

Equation (40), if applied twice, also yields

\[
\left[\left((a_1 \cdot a_2) \cdot a_3 : c\right)\right] = \kappa \left[\left((a_1 \cdot a_2) : c\right)\left(a_3 : c\right)\right] = \kappa^2 \left[\left(a_1 : c\right)\left(a_2 : c\right)\left(a_3 : c\right)\right].
\]

(41)

From (40), it is now evident that

\[
[(b \cdot a) : c] = [(a \cdot b) : c],
\]

(42)

and, from (41), that

\[
\left[\left((a_1 \cdot a_2) \cdot a_3 : c\right)\right] = \left[\left((a_1 \cdot a_2) \cdot a_3 : c\right)\right].
\]

(43)

If one considers §12, No. 2, it follows from (42) that

\[
(b \cdot a) = (a \cdot b)
\]

and from (43) that

\[
(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)
\]

i.e., the commutative and associative laws are satisfied by the new multiplication postulated.\(^{11}\)

It follows from formula (40) and from the final formula in §16 (equation (34)) that the new multiplication must be identical with one of the multiplications defined in §16.

**AUTHOR’S NOTES**


2. I think it indubitable that all developments here considered as "purely logical" are not reducible to logical formalisms as traditionally considered within philosophy or to a ready-made symbolic calculus.

3. While with line segments it must be assumed that they can be compared and necessarily found to be either equal or unequal, one can prove from axioms, in which the word "area" does not occur, and on the basis of certain definitions, that two figures can be compared with respect to area (Schr, Sitzungsberichte der Dtsch Naturforfscher-Gesellschaft, 1892; Killing, *Grundlagen der Geometrie*, Vol. 2, Part 5, §5; Hilbert, *Grundlagen der Geometrie*, 1899, p. 48). This is based on the fact that equality of line segments is a primitive concept and equality of areas in figures is a derived geometrical concept (see my paper *Anschauung und Denken in der Geometrie*, 1900, p. 2, and Zindler, Beiträge zur Theorie der mathematischen Erkenntnist, Sitzungsberichte der phil-hist. Classe der K. Akad. d. Wiss. zu Wien, 1889, Vol. 118, p. 32); Zindler calls primitive concepts “axiomatic”.

4. For convenience, it has been assumed that no equal magnitudes exist which are discernable, that is, non-identical. Consequently, the axioms that two magnitudes are equal when they equal a third magnitude and that equal added to equal results in equals are unnecessary. Of course, these facts must be considered in applications (see axiom (z) in §18).

I intend only to propose a simple system of axioms from which the properties of the ordinary continuum of magnitudes can be derived; I do not intend to establish special kinds of magnitudes as was done by Bettazzi (Teoria delle grandezze, 1800), nor to enlarge upon the ordinary concept of the continuum, as was attempted by Veronese in his “Continuo assoluto” (*Atti della R. Acc. dei Lincei*, ser. 4, memoria d. cl. d. se. f. vol. 6, 1889, p. 613; see also *Fondamenti di geometria a più dimensioni e a più specie di unità rettilinee esposti in forma elementare*, 1891, German translation by Schepp, 1894).

Something further can be said about the independence of the proposed axioms. I will assume from the outset that axioms I, III, and VI obtain. It can then be shown that axioms II, IV, V, and VII are, in a certain sense, independent of each other, as well as independent of the axioms already assumed. More specifically, there is a system of objects for which axiom II does not hold, while axioms I, III, IV, V, VI, and VII are satisfied, namely, all positive integers. There is also, a system of objects for which axiom VII does not hold, while all other axioms are satisfied, namely, all positive rational numbers. A range of objects for which all axioms except IV hold is given by all real, positive and negative (rational and irrational) numbers (numerical magnitudes) together with zero, given an algebraic interpretation of “greater” and “less”. To get another range of objects which fails to satisfy only axiom V, consider all pairs of numbers, \((x, y)\), where \(x\) may be any non-zero positive number and \(y\) any number from \(a\) to \(b\) inclusive, which may be non-zero and positive. Here, \((x, y) > (y, \ y')\), if either \(x > x'\), or when \(x = x'\), \(y > y'\). The sum, \((x, y) + (x', y')\), is defined by the formula \((x + x', y)\), where \(y\) denotes the greater of the numbers \(y\) and \(y'\).

Axioms I to VI, used here, correspond to Veronese’s principles I to III (see *Atti d. Acc. d. Linc., op. cit.*, pp. 604 and 610). Axiom VII is really Dedekind’s axiom of continuity (see axiom (k) in §18 and note 33), which makes our system of magnitudes a continuum, given that axioms I to VI are also satisfied.

Of course, equivalent axiom systems may be postulated. Initially, one could omit the concepts of “greater” and “less”, in which case, as already stated in the text, non-identical magnitudes are taken to be different, and it is only required that:

\[[1]\] Two ordered magnitudes, \(a\) and \(b\), taken in a specific order, have a uniquely defined sum, \(a + b\).

\[[2]\] \(a + b\) differs from \(a\) and from \(b\).

\[[3]\] \((a + b) + c = a + (b + c)\).

\[[4]\] If \(a\) differs from \(b\), then there exists either an \(x\), such that \(a + x = b\), or an \(x\), such that \(b + x = a\).

It is immediately apparent that of the two magnitudes, \(x\) and \(x'\), in [4], only one can exist in any given instance, because otherwise \((a + x) + x' = a\), i.e., \(a + x + x' = a\), which contradicts [2]. One can now stipulate that a
is to be called less than \( b \) whenever \( a + x = b \), and so it is clear that axiom I is satisfied. In this way the concepts "less" and "greater" are presented as defined terms, while in the text they are presented as primitive, i.e., axiomatic, terms.

Not all of the axioms given in the text have yet been satisfied; for instance, only the first parts of IV and V are. If the following is required:

[5] For two magnitudes, \( a \) and \( \delta \), there always exists a third one, \( \gamma \), such that \( \gamma + a = a + \delta \), it is immediately obvious that the second part of axiom V is satisfied. In addition, the second part of IV, which states that \( a - b > 0 \), is now able to be proved. In the first place, consider the fact that according to \([2]\), \( a + b \) differs from \( b \); hence, it must be the case that either \( a < b \) or \( a > b \). If the latter were the case, one would have \( \{a + b\} + x = b \), which would mean, according to \([3]\), that \( \gamma + (a + b) = b \) or \( (\gamma a) + b = b \), which is incompatible with \([2]\). Thus, IV is satisfied and only two more axioms analogous to II and VII are left to be included in some appropriate formulation.

The axiom system could also be altered in a different way, e.g., by retaining axioms I to IV, as well as VI and VII, while replacing axiom V with conditions not mentioning equality but referring only to the concepts "greater" and "less". Additionally, the facts derived in \([2]\). Nos. 1, 2, and 3 will, in this case, be postulated from the outset. If \( a < b \) then all magnitudes can be divided into two classes such that all those which added to \( a \) sum to something \( < b \) belong to the first class, and the second class contains those where the resulting sum \( \geq b \). Having assumed this requirement, it follows that every magnitude in the first class is less than each magnitude of the second class; but it is not yet obvious that the first class is non-empty, and so we must postulate the following:

[6] If \( a < b \) then \( c \) exists such that \( a + c \) is also less than \( b \).

It now follows from axiom VII that there exists a magnitude, \( \zeta \), such that for \( \zeta < \zeta \) the sum of \( a + \zeta \) is less than \( b \), and for \( \zeta > \zeta \) the sum \( a + \zeta \) is greater than \( b \). We can even say, that in the latter case \( a + \zeta = b \) must hold; for if \( a + \zeta = b \) we could find (according to \([3]\) in \([2]\)) \( \zeta \) between \( \zeta \) and \( \zeta \); \( \zeta \), and \( \zeta \) would follow from the last equation that \( a + \zeta \) is greater than \( b \), which, since \( \zeta > \zeta \) would contradict the properties of magnitude \( \zeta \).

Let us not neglect to prove that in fact \( a + \zeta = b \). (In Weber, Lehrbuch der Algebra, 2nd ed., Vol. 1, 1898, pp. 8–9, this has been overlooked.) If \( a + \zeta = b \) then, according to \([6]\), there would exist \( \eta \) such that \( a + \zeta + \eta < b \), i.e., if \( x + (\zeta + \eta) < b \), and this would contradict the properties of magnitude \( \zeta \). But so far it is not obvious that \( a + \zeta > b \) cannot hold. Consequently I introduce the following axiom:

[7] If \( a + b > a \) and \( a + c > a \) then \( a + b > a \).

We were the case that \( a + \zeta > b \), then according to \([6]\) there would be \( \eta \) such that \( a + \zeta + \eta < b \). Since according to \([7]\) there must also be \( \zeta \) such that \( a + \zeta + \eta = b \), it follows (actually indirectly) that \( a + \zeta > b \). Since by assumption \( \zeta < \zeta \), this result contradicts the properties of magnitude \( \zeta \).

That proves the first part of axiom V; proving the second part requires introducing still more axioms.

Hilbert has recently raised the question of the consistency of the axioms of magnitudes (Mathematische Probleme, Göt. Nachr. 1900). Until now the general opinion has been that the consistency of axioms I to VII is shown through the modern development of the theory of (rational and irrational) numbers. See also note 21. From the consistency of the axioms of magnitudes one can derive the consistency of the geometrical axioms (\( x \)) in (\( x \)) of Part II of this work and vice versa (see note 33).

5. It will be shown in \( \$4 \) that the so-called Archimedean axiom follows from Dedekind's axiom of continuity (VII) and the other axioms postulated, while, on the other hand, when one only considers the positive rational numbers, axioms I to VI along with the Archimedean axiom may be satisfied without entailing the truth of the axiom of continuity (VII).

A substantial portion of the results developed in this work remain when only axioms I to VI and the Archimedean axiom are assumed and axiom VII is not postulated.

Hilbert (Jahresbericht der deutschen Mathematiker-Vereinigung, Vol. 8, 1909, p. 180) has, in the context of the other axioms of magnitude, replaced Dedekind's axiom of continuity by two principles: the Archimedean axiom and the "axiom of completeness".


7. The developments of these paragraphs require only axioms I to VI, just as those of previous paragraphs.

8. This axiom (I. der Dedekind's "Was ist und was sollen die natürlichen Zahlen" rec. Heber, vol. 1, 1880, p. 11) is given here as a provable proposition. In the literature there is not the clarity one would desire about its relationship to axiom VII. Stolz (Math. Ann., Vol. 22, p. 510) has noted that the Archimedean axiom is a consequence of continuity if continuity is defined according to Dedekind, i.e., as in axiom VII. This remark is correct if axioms I to VI are assumed as well. The proof given by Stolz (op. cit., p. 511 and in his Vorlesungen über allgemeine Arithmetik, Part 1, 1885, pp. 821) is not sufficient, which I do not think needs to be proved here, since Stolz has withdrawn his remark in response to objections raised by Veronese (Math. Ann., Vol. 39, pp. 107–112).

Veronese (op. cit., p. 612) has claimed that the concept of a continuum must be stated differently from the way it is given by Dedekind, that Dedekind's axiom (our axiom VII) contains the Archimedean axiom, and, furthermore (see Veronese, p. 603), that Stolz's definition of continuity (in his Vorlesungen über Arithmetik, p. 82) assumes the Archimedean axiom and that, consequently, Stolz's proof of this axiom is superfluous.

The comment that the Archimedean axiom is "contained" in Dedekind's axiom of continuity could lead to misunderstandings. I emphasise that the Archimedean axiom can be deduced from axiom VII by the aid of axioms I to VI, but only via the proof given in the text or something similar, which is why such a proof is by no means superfluous.

Of course, the axioms chosen are arbitrary, up to a point, and it is a matter of convenience whether Dedekind's axiom of continuity plus axioms I to VI or some other axioms are to be preferred.

Veronese (op. cit., p. 612, Principle IV) introduces the following postulate as an axiom of continuity: If two magnitudes, \( x \) and \( x' \), vary in such a way that \( x \) always increases, \( x' \) always decreases, that \( x < x' \) remains true, and \( x - x' \) becomes infinitely small, then a magnitude exists in the system which is greater than any value taken by \( x \) and less than any value taken by \( x' \).

If one disregards the dependence introduced between the variables \( x \) and \( x' \) by the supposition of temporal change, then the following is being assumed:

There are two classes of magnitudes, the magnitudes \( x \) and the magnitudes \( x' \); no magnitude can belong to both classes at the same time, which does not necessitate that both classes together comprise the totality of all magnitudes or all magnitudes within an interval. Each magnitude, \( x \), should be less than any magnitude, \( x' \), there should be no greatest magnitude in \( x \) and no least magnitude in \( x' \); and for each magnitude, \( x \), in the totality of all magnitudes, an \( x \) and an \( x' \) can be found such that \( x < x' < x' < x \). Veronese's postulate implies that under these assumptions there exists a magnitude which lies between the two classes and which differs from \( x \) and \( x' \). Moreover, it should be noted that this postulate has at least been modified in its form to the extent of omitting the relationship between \( x \) and \( x' \) put forward by Veronese.

From this postulate, in conjunction with only axioms I to IV, neither the Archimedean axiom nor Dedekind's axiom of continuity follows. However, the latter can be deduced from the (modified) postulate of Veronese if it is conjointed with the Archimedean axiom and axioms I to VI. If one thinks of a division of all magnitudes, as suggested in Dedekind's axiom (VII), \( x \) being a magnitude of the first class, \( y \) a magnitude of the second class, and \( \delta \), any magnitude, then, because of the Archimedean axiom, magnitudes
Discuss the other axioms.

One can therefore find two magnitudes, \( X + (v - 1) \delta \) and \( X + v \delta \), which differ from each other by \( \delta \), and of which \( X + (v - 1) \delta \) belongs to the first class, and \( X + v \delta \), to the second class. Here, \( \delta \) is any arbitrary magnitude. If there were neither a greatest magnitude in the first class nor a least magnitude in the second class, then the conditions of the modified postulate of Veronese would be met, and this would lead to a magnitude belonging to neither class, contrary to the original assumption that all magnitudes are distributed into the two classes. If the first class contained a greatest magnitude, \( x_1 \), and at the same time the second class contained a least magnitude, \( x'_1 \), then these two magnitudes would differ, since as above, in VII, each magnitude is assigned to only one class, and according to §2, No. 3, a magnitude must lie between \( x_1 \) and \( x'_1 \), so it would not belong to either class, which contradicts the terms of the originally assumed division. So the only possibility that remains is that either the first class contains a greatest magnitude, \( x_1 \), and the second class does not contain a least magnitude or vice versa, the second class contains a least magnitude, \( x'_1 \), and the first, no greatest. In the first case the magnitude whose existence is required by Dedekind's axiom (VII) would be \( x_1 \), and in the second case \( x'_1 \). This axiom is therefore fulfilled (see also Veronese, op. cit., p. 613, No. 5a).

Because of Veronese's selection of the axiom of continuity, one is forced to introduce the Archimedean axiom as a special axiom if one wishes to describe the usual continuum (in Ascoli, R. Instituto Lombardo di Sc. e Lett. Rend., ser. II, Vol. 28, 1885, pp. 1060ff. substantially the same formulation of continuity is given as in Veronese without a more detailed discussion of the other axioms).

A system of objects which satisfies Veronese's axiom of continuity, but not the Archimedean axiom nor axiom VII, is obtained in the following way: consider all functions of \( y \) of the form, \( ay + by^2 \), where \( a \) is a positive integer or zero and \( b \) any real (finite) numerical value, but where, when \( a = 0, b \) must be positive and non-zero. If one stipulates that of the two functions, \( a_1 y + b_1 y^2 \) and \( a_2 y + b_2 y^2 \), the first is called the greater when \( (a_1 + b_1 y^2) - (a_2 + b_2 y^2) \) is positive for small positive values of \( y \), and where the less is called the greater when \( b_1 = 0 \), \( a_1 \neq 0 \) to a first class and those where \( b_1 = 0, a_1 = 0 \) to a second.

In this way two classes of values of \( y \) are obtained which again satisfy axioms I to VII must satisfy the commutative law of addition. So it is not proved by Veronese's reasoning (see note 9) that the existence of aliquot parts can be established without the commutative law. That is why I gave priority to the commutative law and proved it without the use of aliquot parts.

One must take into account that the magnitudes established towards the end of note 8 (functions of variable \( y \)), which do not satisfy axiom VII, do not always permit division into \( n \) equal parts, since, for instance, \( 2 \) is not a magnitude of the system.

Euclidean Elements (ed. Heiberg, 1883-88) Book 5, Definition 5. See also the note at the end of §14.

The fact that Euclid tacitly uses the associative law of addition might be explained by the fact that he mainly thinks of stretches where this law is obvious (see §19, No. 3). But he also assumes the commutative law of addition as Veronese (op. cit., p. 621 footnote) has correctly observed. See also No. 1 in Book 5 of Elements.


Even the more recent treatments known to me (see Stolz, Vorlesungen über allgemeine Arithmetik, Vol. 1, 1885, pp. 55ff. 97ff., Lehrbuch der analytischen Geometrie, 1898, Introduction; Weber, Lehrbuch der Algebra, 2nd ed., Vol. 1, 1898, pp. 5–16) deal with the Euclidean theory of proportion without discussing its relationship to the modern, arithmetical theory of irrational numbers, and this is in fact also the case with those authors (as in Stolz and Weber) which deal with both theories. Unfortunately, I was unable to consult the work of Bettazzi, La definizione di proporzione ed il V libro di Euclide (Periodico di Math. VII, pp. 16ff. & 54), but it does not seem to have the same aim as I plan to follow in this paper (see Fortschritte der Mathematik, 1892, p. 501).

A similar supplementation is also required to the proofs of "projec- tive measures" given to date in recent geometry.

In the first place, the sum of two magnitudes has to be conceived of as something entirely different from an arithmetic sum. For example, if the magnitudes are line segments, then their addition, the concatenation of the line segments, is to be understood geometrically.

The three purely arithmetical forms of the definition of numbers which have been used by the above authors have been collected by G. Cantor, Grundlagen einer allgemeinen Mannigfaltigkeitslehre, 1883, 21ff.

The totality of the lower numbers of a cut is what Pasch (Einleitung in die Differential- und Integralrechung, 1882, p. 3) calls a "Zahlen- strecke." 46

In particular, it can be proved in a purely arithmetical way, that is, without assuming any axioms, that axioms I to VI, along with the Archimedean axiom, are fulfilled for cuts (see Dedekind, op. cit.; Pasch, op. cit.; Tannery, Introduction à la théorie des fonctions d'une variable, 1886, pp. 1ff; Weber, Lehrbuch der Algebra, 2nd ed., Vol. I, 1898, pp. 5ff). If one conceives of the totality of all cuts as completely given and if one thinks of
a division of all cuts into two classes, such that the classes are of the kind assumed in axiom VII, then one can easily construct a cut having the properties of magnitude $\xi$ required in axiom VII; therefore, axiom VII seems fulfilled for cuts. Still, one might have some doubts about saying that the totality of all cuts constitutes a completely given totality which satisfies axioms I to VII, for it should not be forgotten that a cut is nothing more than a special rule dividing all rational numbers; and one may well be unable to conceive of the totality of all such rules. One could therefore take the view that in postulating the continuum of numbers one makes another assumption, an assumption which may not be motivated by purely arithmetic considerations but, rather, by one's external experience, that is to say, by such a notion as $a \div b$. If one only knows that magnitudes are a continuous set, with suitably chosen integers, $\nu$ and $\mu$, fulfilling equation $v=\mu b$, then one can prove the existence of a magnitude, $c$, of the required kind. This is quite easy if one employs the existence of aliquot parts. But one does not necessarily need to use aliquot parts, the existence of which, in our development, rested on the axiom of continuity. One can also deduce magnitude $c$ from magnitudes $a$ and $b$ via a process analogous to that of the greatest common factor of numbers (Euclid's Elements, Book 7, No. II and Book 10, No. III); then one only needs to postulate axioms I to VI.

If a magnitude, $c$, is given, such that $a=\mu c$, $b=\nu c$, and also a magnitude, $\mu$ or $\nu$, in such a way that $a'c'=b'c'$, it follows from the developments in the text that $\mu'-\mu=\nu+c'(c'c)$, see also, my publication: Anschauung und Denken in der Geometrie, pp. 47f.).

25. See Euclid, Elements, Book 10, No. V.

26. Euclid's Elements, Book 10, No. VI.

27. One could always define the cut corresponding to a ratio, $a/ b$, even if only axioms I to VI are assumed, without either the axiom of continuity or the Archimedean axiom, if one allows only (which has not been done here) that all rational numbers may fall into the first class of the cut or that all rational numbers may fall into the second class of the cut, so that one class remains completely empty. With this interpretation there will be a cut corresponding to the number, 0, and one corresponding to the number, $\infty$. In a certain sense, the relations (19) and (21) would remain; but one could no longer conclude that $a_1=\mu_1 a_2$ from $\{a_1, b_2\} = \{a_2, b_1\}$, when for instance $\{a_1, b_1\} = 0$, and $\{a_2, b_2\}$ (see the magnitudes towards the end of note 8). Euclid's definition of proportion (Elements, Book 5, Definition 5) can be restated in the following way: one says that $a$ is exactly related to $b$ in the same ratio as $c$ to $d$ if for every integer, $\mu$, and every integer, $\nu$, the multiple, $\mu c$, of $a$ is less than, equal to, or greater than the multiple, $\nu b$, of $b$, just when the corresponding multiple, $\mu c$, is less than, equal to, or greater than the corresponding multiple, $\nu d$, of $d$. When one possesses Dedekind's concept of cut, the ratio, $a/ b$, holding between two magnitudes, $a$ and $b$, is easily grasped as something independent by coordinating a cut with this ratio, and it is then immediately apparent that the Euclidean definition of proportion is synonymous with this explication: $a$ and $b$ are in the same ratio as $c$ and $d$ whenever the cut corresponding to the ratio, $a/ b$, is identical with that corresponding to the ratio, $c/ d$. This close relationship between the concept of cut and the Euclidean concept of proportion was recognised by Dedekind (Was sind und was sollen die Zahlen?, 1888, foreword, p. xiv).

29. If one has two systems of magnitudes, both fulfilling axioms I to VII, then the systems can be explicitly related to each other such that the sums of corresponding magnitudes also correspond (relations such as these have been examined by Betzuzzi, see note 16).

To do this one only has to select a unit within each system and bring into correspondence those magnitudes having the same measure-number.

30. Geometrical and physical magnitudes, too, such as line segments, areas, times, masses, forces, etc., as given in the first place, can only be added and addition is only possible with magnitudes of the same kind. Talk of the multiplication of geometrical or physical magnitudes is based upon arbitrary convention.

The present paper assumes only such axioms as relate to the comparison and addition of magnitudes. These axioms are sufficient to deduce the Euclidean theory of proportion and the contemporary theory of measurement. The results concerning the multiplication of magnitudes appear here only as corollaries to one of these theories.

Up to this point this section (see also notes 7, 9, and 22) has only used those earlier results which follow from axioms I to VI and the Archimedean axiom. One can, therefore, if one assumes axioms I to VI, the Archimedean axiom, and both distributive laws (35) and (36), already validly infer the commutative and associative laws of multiplication. This result contains a more specific consequence of Hilbert (Grundlagen der Geometrie, 1899, pp. 72f.).

TRANSLATORS' NOTES

1. Hölker's term Mass presents the English translator with significant problems. Literally, the nearest English equivalent is (the) Measure, as in r is the measure of a in units of b (where r is a real number; a, a magnitude of some quantity (like a particular length); and, b, a unit of the same quantity (like the length of the standard metre)). Thus, the theory referred to in Hölder's title is that of numerical measures of quantities. Considering no more than this, the most apt translation would be measure.

However, since Hölder's time, the term measure has acquired other connotations for the mathematician (as in measure theory) and the meaning of the term measurement has broadened within the mathematical literature so that the expression measurement theory now includes what Hölder meant by die Lehre vom Mass. Hence, the translator can avoid some ambiguity and conform to a relevant modern usage by translating Mass as measurement.

This is not a perfect solution because measurement has many meanings. Among them are the process of measuring, the numerical result obtained via that process, and the more theoretical sense of the numerical measure of a quantity (which may or may not be obtained via measurement in the first sense and may or may not be the same numerical value as the measurement in the second sense). It is this more theoretical sense of measurement that covers the meaning of Hölder's Mass. Because most readers of this translation will be very familiar with this last sense, translating Mass as measurement remains the preferred option.

2. The German term Grösse has been consistently translated here as magnitude, although quantity is another possible translation of that term. As Hölker distinguishes Größe from Quantität and the latter is intended as the more general concept of the two, we have maintained his distinction via the English terms, magnitude and quantity.

3. Hölder's term ganze Zahl is literally whole number. We have consistently translated it as integer.

4. With the concept of procedure is the most direct translation of Hölder's mit dem Begriff des Verfahrens, but another plausible translation would be with the concept of the (this) procedure, the reference to the particular correspondence corresponding to each assumption in the German being understood.

5. This is a reference to the Kantian technical, philosophical concept Anschauung, the standard English translation of which is intuition. Very
generally, according to Kant, intuitions are the objects of experience as immediately “given” to us and three dimensional Euclidean space is one of the \textit{a priori} forms of intuition (i.e., one of the ways in which the mind structures such experience). Hence, the axioms of geometry, for example, were held to be derived from the faculty of intuition (see Caygill (1995) and Editor (1891–92)). The Kantian epistemology of mathematics was not only a matter of considerable controversy around Hölder’s time, it was a controversy to which he contributed (Hölder, 1900), joining with Helmholtz and others against the strict Kantian view.

6. We have translated Hölder’s German term \textit{Strecken} as line segments. In Part II, he uses the same term to refer to a narrower concept, viz., \textit{bewegte Strecken} or, as Russell (1903, p. 181) more economically translates it, \textit{stretches}.

7. Hölder’s term \textit{Inhalt} literally means contents and, so, is more abstract than either of the English terms \textit{area or volume} (\textit{Flächeninhalt} and \textit{Rauminhalt} in German, respectively). Since contents is not a specifically quantitative concept in English, translating \textit{Inhalt} as contents would cause Hölder’s term to lose an essential ingredient. Spatial magnitude might be a better choice, but there is evidence (Author’s Note 3, where he obviously Holders term to lose an essential ingredient.

8. We have translated Hölder’s special term Masszahlen here is translated as measure-number, following Niederer’s (1992, p. 245) suggestion.

9. Hölder’s expression \textit{a verhält sich zu b wie zu d} is impossible to translate literally because there is no precise English equivalent. One suggested translation is \textit{a relates to b as c to d}. This is not optimal because the precise relation that Hölder intends is not specified. He intends the relation characterized by Euclid in Dfn. 5, Bk. V of \textit{The Elements}:

Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second or fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order. (Heath, 1908, p. 114).

That is, the relation \textit{in relation to b is that of the ratio of a to b}. The ideal translation would be via an English verb, \textit{R}, capturing this relation and of the form: \textit{a R to b as c to d}. There is no English verb that does the trick. Herland (1951, p. 208) proposes that a similar expression, \textit{16 verhält sich zu 32 wie zu 2}, be translated as, \textit{16 is to 32 in the ratio of 1 to 2}, the analogue of this in Hölder’s case then being \textit{a is to b in the ratio of c to d}. Alternatively, one could equivalently say, a \textit{relates to b in the same ratio as c to d}. This has the advantages of (a) staying close to Hölder’s form of words and (b) specifying the relation that Hölder intends to identify. We have used this location consistently for the above and cognate expressions.

10. Here Hölder uses the noun \textit{Verhältnis}, which explicitly means ratio.

11. Again, it is the German \textit{Masses} or measurement in the sense of the measure (see 1 above).

12. It has been suggested to us that Hölder’s term \textit{zugeordnete} could have been translated as assigned to, in preference to corresponding to. We have rejected this suggestion for three reasons. First, scientific dictionaries (e.g., Herland (1951, p. 231) and Dorian (1970, p. 869)) do not recommend assigned, but corresponding. Second, assignment is a ternary relation \( (x \; \text{must be assigned to} \; y \; \text{by a third term,} \; z) \) and it is clear from Hölder’s development that it is to a binary relation that he draws our attention. Correspondence, being binary, covers this meaning more adequately. Third, independently of Hölder, the view that measurement involves numerical assignments became widespread in psychology following Stevens (1946). To avoid inevitable misinterpretations, the word assignment is therefore best avoided in this context.

13. Newton’s Latin is translated as, “\textit{By number we understand not so much a multitude of Unities, as the abstracted ratio of any Quantity to another Quantity of the same kind, which we take for Unity}” by Whiteside (1967, p. 2).

14. \textit{Zahlenstrecke} could be translated as a stretch, expanse, or array of numbers.

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Received: May 17, 1994