Representability of binary relations through fuzzy numbers

María J. Campión\textsuperscript{a}, Juan C. Candeal\textsuperscript{b}, Esteban Induráin\textsuperscript{c,}\textsuperscript{*}

\textsuperscript{a}Departamento de Matemática e Informática, Universidad Pública de Navarra, Campus Arrosadía, E-31006 Pamplona, Spain
\textsuperscript{b}Departamento de Análisis Económico, Facultad de Ciencias Económicas y Empresariales, Universidad de Zaragoza.
\textsuperscript{c}Doctor Cerrada 1-3, E-50005 Zaragoza, Spain

Received 21 March 2005; received in revised form 16 June 2005; accepted 20 June 2005
Available online 26 July 2005

Abstract

We analyse the representability of different classes of binary relations on a set by means of suitable fuzzy numbers. In particular, we show that symmetric triangular fuzzy numbers can be considered as the best codomain to represent interval orders. We also pay attention to the representability of other classes of acyclic binary relations. © 2005 Elsevier B.V. All rights reserved.

Keywords: Fuzzy numbers; Preference modelling; Decision analysis

1. Introduction

In the present paper we address the problem of the numerical representability of binary relations, or orderings. The mathematical statement of the underlying problem is quite simple (see e.g. [11, p. 200]): We are given a nonempty set $X$ with a binary relation $\mathcal{R}$. We are looking for order-preserving functions $F : X \rightarrow \mathbb{R}$, accomplishing, say, that $x \mathcal{R} y \iff F(x) \leq F(y)$ ($x, y \in X$). Surveys on the numerical representations of ordinal structures continue to appear (see e.g. [16] or [31]; for more recent discussions one should consult the literature between mathematics and theoretical computer science such as e.g. [30] or [31]).

\textsuperscript{*}Corresponding author. Tel.: +34 948 169551; fax: +34 948 169521.
\textit{E-mail addresses:} mjesus.campion@unavarra.es (M.J. Campión), candeal@posta.unizar.es (J.C. Candeal), steiner@unavarra.es (E. Induráin).

0165-0114/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.fss.2005.06.018
There is some remarkable interdisciplinary aspect in this issue. On the one hand, this kind of results is of particular importance in mathematical economics related to constant returns to scale economies (see, e.g., [33, Chapter 2]). Also, measurements that are often encountered in the social or biological sciences (see e.g. [43, 35] or [46]) are usually based on data that can be “compared” but not a priori “quantified”. One obtains a scale but not a yardstick. All quantification a posteriori is based on the hypothetical possibility to map the set of data into the set $\mathbb{R}$ of real numbers under preservation of order. In the economic and social science contexts such a mapping is called a utility function. It is desirable that the mathematic community be made aware of the mathematical needs of, say, the social science community in this area.

It is undeniable that the main emphasis in the utility theory literature, which deals with the study of order-preserving numerical representations of binary relations (mainly preordered sets), has been on real-valued functions (see e.g. [38]). It may be that the almost exclusive use of $\mathbb{R}$ as the codomain of order-preserving representations (also known as utility functions) is based on the implicit argument that, intuitively, “utility” is a (real) number. As a matter of fact, it is clear that most economists do employ the concept in this manner. In addition, it is, perhaps, also implicitly accepted that real-valued utility functions are used because $\mathbb{R}$ is mathematically much simpler than other possible totally ordered codomains. However, as recently proved and extensively discussed in [32], “it is highly desirable and even imperative to begin the development of a theory of the existence and continuity of non-real-valued utility functions”.

In our opinion, we can interpret such an assertion in a wide sense. On the one hand, we could continue working in the way initiated in that paper [32] and consider codomains quite different from the real line $\mathbb{R}$, as, for instance, the lexicographic plane or the long line (see e.g. [8]): in this direction the utility functions are, obviously, “non-real-valued”.

On the other hand, we could also follow a new path, studying order-preserving representations consisting of utility functions that are still “real-valued” in some sense, but take values in a suitable family of fuzzy real numbers. These families of fuzzy numbers will be given some ordering that, in general, is not isotonic (order-isomorphic) to the usual total order $\leq$ of the real line $\mathbb{R}$.

There are some important motivations, all of them related, so to proceed:

1. First, we may notice that given a nonempty set $X$, endowed with a binary relation $\mathcal{R}$, the mere existence of a map $F : X \to \mathbb{R}$ such that $x R y \iff F(x) \leq F(y)$ ($x, y \in X$) imposes strong restrictive conditions on $\mathcal{R}$. Indeed, in this case $\mathcal{R}$ must be a total preorder (i.e., reflexive, transitive, and complete). There are, however, many contexts in which such conditions cannot be accomplished. For instance, in Psychology and Economic Theory it is common to deal with models of preferences that fail to be transitive (see [14, 24, 25, 27, 29, 50]). The following example concerning decision theory illustrates the fact that a preference relation $\mathcal{R}$ may fail to be transitive, where a preference is understood here in the “weak sense” (i.e., $x R y$ means “the element $x$ is at least as good as the element $y$”):

Consider a total preorder $\mathcal{R}$ on a real cone $K$ of nonnegative real random variables (i.e., measurable real functions) on a probability space $(\Omega, A, P)$. Elements of $K$ are interpreted as random gains. Assume that $K$ is endowed with the induced topology $\tau_K$ of any vector topology $\tau$ on the space of all the real random variables on $(\Omega, A, P)$, and that $\mathcal{R}$ is homothetic (i.e., $x R y \iff tx R ty$ for every $x, y \in K$, and $t \in (0, +\infty)$) and continuous (i.e., $\{z \in K : z R x\}$ and $\{z \in K : x R z\}$ are closed sets for every $x \in K$). Let $x \in K$ be such that $x = 10^6P$-almost surely. For any individual, $x$ is strictly preferred to the constant $\hat{0}$. For many individuals, $x$ and $tx$, with $t = 1 - 10^{-6}$, are indifferent (i.e., $tx R x$ and $x R tx$). If this happens, then $x$ and $t^nx$ are indifferent for every $n \geq 1$ by homotheticity and
transitivity of $\mathcal{R}$, so that $x$ and $\bar{0}$ should be judged indifferent since $\mathcal{R}$ is continuous. So, the assumption of nontransitivity of the binary relation $\mathcal{R}$ seems to be more realistic, because otherwise a too accurate assessment of the preferences is required.

This example is similar to a classical one introduced in [37]:

Find a subject who prefers a cup of coffee with sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with $(1 + \frac{1}{100})x$ grams of sugar, $i = 0, 1, \ldots, 400$, where $x$ is the weight of one cube of sugar. It is evident that he will be indifferent between cup $i$ and cup $i + 1$, for any $i$, but by choice he is not indifferent between $i = 0$ and $i = 400$.

Also, non-transitive preferences may appear in a social choice setting, fusing the preferences of multiple agents with respect to a set of alternatives, in the search for a “social” preference that is acceptable to all agents. Even in the case in which the individual agents report total preorders, it may happen that the aggregated social preference is not even transitive. Following Chapter 6 in [44], by “aggregation” we generally understand a mechanism which associates some structure to any family of structures of a given type (preferences, for instance) in a sensible manner. What needs to be done is to construct a common aggregated preference following several criteria that are acceptable for all agents. Of course, what is “sensible” takes different forms according to the criteria that could appear in each context. It is well known that this general problem could lead to notorious impossibility theorems (see e.g. [6]). An easy example where intransitivity appears after aggregation is the following one:

Consider several car-sellers as individual agents that rank different cars through the price to which they sell them. Of course, different car-sellers could report different prices to sell a similar car. The list of prices furnished by a car-seller is a numerical representation of his “preferences”. A potential buyer will act “aggregating” the prices encountered in the car market, that is, he would assign to each car not necessarily a single price, but the interval of all possible prices to which such car could be sold among the different car-sellers. Thus, assume again that a car $A$ could be sold at prices that vary in an interval $[m_A, M_A]$, where $m_A$ is the minimum price of the car $A$ among the car-sellers, and $M_A$ is its maximum price. Similarly the car $B$ could be sold at prices varying in an interval $[m_B, M_B]$. Then, the buyer will consider that “the car $A$ is not more expensive than the car $B$” if $m_A \leq M_B$. In general, this “aggregated relation” defined by the buyer is not transitive.

More particular situations in social choice in which intransitivity of preference is involved may arise in different contexts. For instance, in [9] the motivation is the construction of formalism (languages) to express vague human preferences that are usually expressed in the classical first order propositional logic. In [10] the motivation is multiple software agents reporting their own preference orderings in the spirit of Distributed Artificial Intelligence.

2. We have already mentioned that, given a nonempty set $X$ with a binary relation $\mathcal{R}$, the representability of $\mathcal{R}$ through an order-preserving function $F : X \rightarrow \mathbb{R}$ accomplishing that $x \mathcal{R} y \iff F(x) \leq F(y)$ ($x, y \in X$) forces $\mathcal{R}$ to be a total preorder. This is too restrictive. As a matter of fact, several different kinds of “numerical representations” for binary relations have been introduced in the literature to feature situations in which the binary relation of our interest may fail to be a total preorder. Among those alternative representations, we can consider the following ones:

(i) (Cantor, 1895 [19]): Existence of a map $f : X \rightarrow \mathbb{R}$ such that $x \mathcal{R} y \iff f(x) \leq f(y)$ ($x, y \in X$).
(ii) (Wiener, 1914 [51]): Existence of two maps $f, g : X \rightarrow \mathbb{R}$ such that $x \mathcal{R} y \iff f(x) \leq g(y)$ ($x, y \in X$).
We may observe that all these representations deal with functions or correspondences whose codomain construction of a total order (let alone a well-order) any uncountable set (say, the real numbers) is known. Furthermore, there are sets for which Zermelo’s theorem assures that every set can be well-ordered, no specific construction for well-ordering is needed. However, the precise definitions may vary (compare, e.g., [21], [34, Chapter 1] and [40, Section 2.5]). Since we are trying to represent binary relations that usually come from a sort of “ordering” could look like the binary relation to be represented (not necessarily being a total preorder).

(iii) (Swistak, 1980 [48]): Existence of two maps \( f, g : X \rightarrow \mathbb{R} \) such that \( x R y \iff f(x) < g(y) \) \((x, y \in X)\).

(iv) (Riguet, 1951 [45]): Existence of a bivariate map \( F : X \times X \rightarrow \mathbb{R} \) such that \( x R y \iff F(x, y) \geq 0 \) \((x, y \in X)\), or other similar conditions.

(v) (Armstrong, 1950 [5]): Existence of a map \( f : X \rightarrow \mathbb{R} \) and a strictly positive real constant \( k > 0 \) or “threshold” such that \( x R y \iff f(x) \leq f(y) + k \) \((x, y \in X)\).

(vi) (Fechner, 1860 [23], Agaev and Aleskerov, 1993 [1]): Existence of a suitable bivariate map \( F : X \times X \rightarrow \mathbb{R} \), in a way that \( x R y \iff f(x) \leq f(y) + F(x, y) \) \((x, y \in X)\).

(vii) (Fishburn, 1973 [27]): Existence of a set-valued correspondence \( S \) from \( X \) into \( \mathbb{R} \) such that \( S(x) \) is a nonempty bounded subset, for every \( x \in X \), and it holds that \( -(y R x) \iff S(x) \subset S(y) \), and, in addition, \( \sup S(x) < \sup S(y) \) \((x, y \in X)\).

(viii) (Levin, 1983 [36]): Existence of a measure \( \mu \) defined on \( X \), and a set-valued correspondence \( F \) from \( X \) into \( \mathbb{R} \) such that for every \( x \in X \) the set \( F(x) \) is \( \mu \)-measurable, and also \( x R y \iff \mu(F(x)) \leq \mu(F(y)) \) \((x, y \in X)\), where \( \mu(F(x)) \) must be a real number (i.e., finite) for every \( x \in X \).

(ix) (Shafer, 1974 [47]): Existence of a hemisymmetric bivariate map \( k \) that is a map from \( X \times X \) into \( \mathbb{R} \) such that \( k(x, y) = -k(y, x) \) \((x, y \in X)\) \( k : X \times X \rightarrow \mathbb{R} \) such that \( x R y \iff k(x, y) \geq 0 \) \((x, y \in X)\).

(x) (Arrow and Hahn, 1971 [7]): Existence of a metric \( d \) on \( X \) and a subset \( C \subset X \) such that \( x R y \iff d(x, C) \leq d(y, C) \) \((x, y \in X)\).

We may observe that all these representations deal with functions or correspondences whose codomain is the set of real numbers \( \mathbb{R} \).

It seems natural now to think in some alternative codomain that, in some sense also consists of “real numbers”. In this direction, a suitable alternative codomain could be the set of fuzzy real numbers. Basically a fuzzy real number can be understood as a map \( F : \mathbb{R} \rightarrow [0, 1] \) satisfying some “axioms” or properties imposed a priori. However, the precise definitions may vary (compare, e.g., [21], [34, Chapter 1] and [40, Section 2.5]). Since we are trying to represent binary relations that usually come from a sort of “ordering” on the given set \( X \), it is desirable to deal with a codomain that also has defined an ordering with good properties. If we use the real numbers as a codomain, we are thinking, at least implicitly, in a totally ordered set, and, as aforementioned, some other totally ordered codomains have also been introduced in the literature (see [32]).

Constructions where the codomain is some particular class of fuzzy numbers are perhaps not so common. The reason, obviously, is that they are not a totally ordered set. As a matter of fact, though Zermelo’s theorem assures that every set can be well-ordered, no specific construction for well-ordering any uncountable set (say, the real numbers) is known. Furthermore, there are sets for which no specific construction of a total order (let alone a well-order) is known, for example, the set of real-valued functions of one real variable. (See [22, p. 35]). The same happens with the maps \( F : \mathbb{R} \rightarrow [0, 1] \), so that we cannot expect, a priori, to have a total order on the set of fuzzy numbers.

At this point, however, we should realize that we do not need to have always a totally ordered set in the codomain! This depends on the kind of binary relation we are trying to represent, and, as previously discussed, not all of them will be total preorders, so that we can use some new sort of codomain whose “ordering” could look like the binary relation to be represented (not necessarily being a total preorder).

3. Some special families of fuzzy numbers can be given an ordinal structure that is idoneous to represent particular kinds of binary relations on a nonempty set \( X \). Despite through the present paper we are thinking...
on ordinary (i.e., non-fuzzy!) binary relations defined on an ordinary set \(X\), some particular binary relations have associated, at least from a psychological point of view, an idea of “vagueness” or “uncertainty” (not to say “fuzziness”!). A very important example here is that of interval orders. The concept of an interval order was introduced by Fishburn [24] in contexts of Economic Theory, in order to take account of the vagueness of individual preferences when modelling the consumer behaviour, and to study models of preferences whose associated indifference may fail to be transitive (see also [25], or [14]). A very complete and informative study of interval orders appears in Chapter 6 of [16]. We recall that an interval order \(R\) is a reflexive binary relation on \(X\) such that \((x R z)\) and \((y R w)\) \(\Rightarrow\) \((x R w)\) or \((y R z)\) for every \(x, y, z, w \in X\). An interval order \(R\) is not transitive in general. An interval order \(R\) defined on a set \(X\) is representable if there exists a pair of real-valued functions \(u, v: X \rightarrow \mathbb{R}\) such that \(x R y \iff u(x) \leq v(y)\) \((x, y \in X)\).

Roughly speaking, in a representable interval order each point \(x\) is not assigned, in general, a (unique) real number, but instead an “interval of confidence” \([u(x), v(x)] \subset \mathbb{R}\) that reflects the uncertainty, vagueness, or difficulty of the agent (e.g., an economic consumer) to make precise a value. Now we can observe that the concept of uncertain or fuzzy numbers may be presented (see e.g. [34]) to be an extension of the concept of the interval of confidence, which is familiar to anyone who has computed using imprecise data in simple or complex systems (see e.g. [2,4,17,39,42,49]). We note that, in the most usual definitions of a fuzzy number, it happens that the supports of fuzzy real numbers are intervals of the real line \(\mathbb{R}\), and thus it may well be that the theory of interval orders can be interpreted in terms of fuzzy numbers.

4. Fuzzy numbers are not only an alternative codomain of maps used to represent binary relations in some “new manner”. As in the case of interval orders, it is important to point out here that some other of the classical approaches to represent some particular kinds of binary relations on a set also have an equivalent approach based on constructions that use some families of fuzzy numbers as codomain of key maps. In addition, and coming back to total preorders, it also happens that some classical total order that is not representable in the real line \(\mathbb{R}\) with its usual order \(\leq\) can still be represented in some set of fuzzy numbers. An example of this situation is the lexicographic plane \((\mathbb{R}^2, \preceq_L)\) where \((a, b) \preceq_L (c, d)\) if and only if \(a < c\) or \(a = c, b \leq d\). We shall give some other examples through the paper.

As analysed in [8] there are three different classes of non-representable totally ordered sets. These classes are: (i) long chains, i.e. chains containing a subchain isotonic to the first uncountable ordinal, (ii) Aronszajn-like chains, (iii) planar chains, i.e. chains containing a subchain isotonic to a non-representable part of the lexicographic plane. Except the class of planar chains, the other classes depend on set-theoretical axioms or constructions, as the structure of a well-ordered uncountable set. Roughly speaking, the kind of orderings that appear in those classes is much less intuitive than, say, the Euclidean order \(\leq\) of the real line \(\mathbb{R}\). However, interesting classical examples of totally ordered sets that belong to the third class (so that they are non-representable!), namely the lexicographic plane \((\mathbb{R}^2, \preceq_L)\) and the lexicographic spaces \((\mathbb{R}^n, \preceq_L)\) (defined for \(n \geq 3\) by \((a_1, \ldots, a_1) \preceq_L (b_1, \ldots, b_n)\) if and only if \(a_1 < b_1\), or there exist \(k \leq n\) such that \(a_i = b_i\) \((i = 1, \ldots, k - 1)\), \(a_k < b_k\), or \(a_i = b_i\) \((i = 1, \ldots, n - 1)\), \(a_n \leq b_n\)) may still be analysed and interpreted by means of some set (ad hoc) of fuzzy numbers, endowed with a suitable ordering.

2. Notation and preliminaries

In what follows \(X\) will denote a nonempty set, and \(R\) a binary relation defined on \(X\).
The binary relation $R$ is said to be:

(i) A preorder if it is reflexive and transitive. If in addition it is total, $R$ is said to be a total preorder. The asymmetric part $P$ of a preorder $R$ is defined as $xP y \iff (xR y)$ and $\neg(yRx))$ (or $x, y \in X$) and the symmetric part $I$ is defined by $xI y \iff (xR y)$ and $yRx)$ (or $x, y \in X$). In some contexts arising in Economics and Social Sciences, the preorder $R$ is said to be a “preference” (or a “weak preference”), whereas $P$ and $I$ are said to be, respectively, the “strict preference” and the “indifference” associated with $R$. An antisymmetric total preorder is called a total order.

(ii) An acyclic binary relation if for every natural number $n \geq 1$ and any elements $x_1, x_2, \ldots, x_n \in X$ such that $x_i R x_{i+1}$ $(i = 1, \ldots, n - 1)$ it never happens that $x_n R x_1$. (In other words: $x_i R x_{i+1}$ $(i = 1, \ldots, n - 1) \Rightarrow \neg(x_n R x_1)$). Observe that, in particular, any acyclic binary relation is asymmetric and irreflexive.

(iii) An interval order if it is reflexive and $(xRz)$ and $(yRw)$ $\Rightarrow (xRw)$ or $(yRz)$ for every $x, y, z, w \in X$. This implies, in particular, that $R$ is total, since given $x, y \in X$ we have by reflexivity that $(xRz)$ and $(yRx)$ which, by hypothesis, implies that $(xRy)$ or $(yRx)$.

(iv) A semiorder [37] if it is an interval order and, in addition, for every $x, y, z \in X$ it holds that $(xRy)$ and $(yRz)$ $\Rightarrow (xRw)$ or $(wRz)$ for every $w \in X$.

Observe that $R$ is a total preorder $\Rightarrow R$ is a semiorder $\Rightarrow R$ is an interval order, but the converses are not true in general (see e.g. [18]).

A total preorder $R$ defined on $X$ is said to be representable if there exists a real-valued map $F : X \rightarrow \mathbb{R}$ (usually called “utility function”) such that $xRy \iff F(x) \leq F(y)$ (or $x, y \in X$). It is well-known (see e.g. [11, p. 200] or [16, Chapter 1]) that a total preorder $R$ is representable if and only if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $xRy$ there exists an element $d$ in $D$ such that $xRdRy$. The subset $D$ is said to be order dense in $X$ with respect to $R$.

As defined before for the case of preorders, the asymmetric part $P$ of an interval order $R$ is defined as $xPy \iff (xRy)$ and $\neg(yRx))$ (or $x, y \in X$) and the symmetric part $I$ is defined by $xIy \iff (xRy)$ and $yRx)$ (or $x, y \in X$).

An interval order $R$ is said to be representable if there exists a pair of real-valued functions $F, G : X \rightarrow \mathbb{R}$ such that $xRy \iff F(x) \leq G(y)$ (or $x, y \in X$). A semiorder $R$ is said to be representable if there exists a real-valued function $F : X \rightarrow \mathbb{R}$ and a nonnegative real number $k \geq 0$ (also called “threshold”) such that $xRy \iff F(x) \leq F(y) + k$, $(x, y \in X)$.

Following [24,26] we shall associate with an interval order $R$ two new binary relations, each of them being a total preorder, respectively, denoted by $R^*$ and $R^{**}$, that are defined by $xR^*y \iff (yPz \Rightarrow xPz \forall z \in X)$ and similarly $xR^{**}y \iff \neg(yRx), xP^{**}y \iff \neg(yRx), xI^{**}y \iff \neg(yRx), xR^{**}y \iff \neg(yRx)$ and $xI^{**}y \iff \neg(yRx)$ (or $x, y \in X$). Now it is straightforward to see that $xR^{**}y \iff xPzRy$ for some $z \in X$. Also, similarly, $xR^{**}y \iff xRzPy$ for some $z \in X$.

It is straightforward to see that if $R$ is an interval order, then it is transitive (hence it is a total preorder) if and only if $R$, $R^*$, and $R^{**}$ coincide (see e.g. [12]).

A key concept, used in [41] to get a characterization of the representability of interval orders, is that of interval order separability (henceforward i.o.-separability). An interval order $R$ on a set $X$ is said to be i.o.-separable if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $xPy$ there exists an element $d$ in $D$ such that $xPdR^{**}y$. 

The representability of interval orders is characterized as follows:

**Proposition 1** (see Olóriz et al. [41]). Let $X$ be a nonempty set endowed with an interval order $\mathcal{R}$. Then, the following statements are equivalent:

(i) $\mathcal{R}$ is i.o.-separable,

(ii) there exists a bivariate map $F : X \times X \to \mathbb{R}$ such that $x \mathcal{R} y \iff F(x, y) \geq 0$ and $F(x, y) + F(y, z) = F(x, z) + F(y, y)$ for every $x, y, z \in X$,

(iii) $\mathcal{R}$ is representable as an interval order.

Some other characterizations of the representability of interval orders had already appeared in [20,27,29]. In what concerns semiorders, a (difficult!) characterization of the representability of semiorders has recently been achieved in [18]. In particular, as in part (ii) Proposition 1, it can be proved that the representable semiorders furnish a solution of certain functional equation, namely:

The following assertions are equivalent for a nonempty set $X$ endowed with a semiorder $\mathcal{R}$:

(i) There exists a bivariate map $F : X \times X \to \mathbb{R}$ such that $x \mathcal{R} y \iff F(x, y) \geq 0$ and $F(x, y) + F(y, z) = F(x, z) + F(t, t)$ for every $x, y, z, t \in X$,

(ii) $\mathcal{R}$ is representable as a semiorder.

In what concerns fuzzy sets and fuzzy numbers we shall follow the definitions and notations that appear in [40]. Thus, $U$ being a set (usually called universe) and $\mathcal{L}$ being a lattice with greatest element $\bar{1}$ and smallest element $\underline{0}$, a fuzzy set is a function $A : U \to \mathcal{L}$. The support of $A$ is the set $\text{Supp}(A) = \{x \in U : A(x) \neq \underline{0}\}$, and the kernel of the fuzzy set $A$ is the set $\text{Ker}A = \{x \in U : A(x) = \bar{1}\}$. A fuzzy set $A$ is said to be normal if it has nonempty kernel. Given $\alpha \in \mathcal{L}$, the $\alpha$-cut of the fuzzy set $A$ is the set $A_\alpha = \{x \in U : A(x) \wedge \alpha = \alpha\}$, where $\wedge$ denotes the latticial operation in $\mathcal{L}$. In the particular case in which $U = \mathbb{R}$, $\mathcal{L} = [0, 1] \subseteq \mathbb{R}$ and $x \wedge y = \min\{x, y\}$, the fuzzy set $A$ is said to be convex if for every $\alpha \in [0, 1]$ the $\alpha$-cut $A_\alpha$ is a convex subset of $\mathbb{R}$.

In this case, a fuzzy number is a normal convex fuzzy set such that the function $A : \mathbb{R} \to [0, 1]$ is piecewise continuous, and there exist points $a_1 \leq a_0 \leq b_1 \leq b_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$ with the following properties:

(a) $a_1 \in \mathbb{R} \cup \{-\infty\}$ and $b_1 \in \mathbb{R} \cup \{+\infty\}$,

(b) $a_0, b_0 \in \mathbb{R}$,

(c) $\text{Supp}(A) \subseteq [a_1, b_1] \cap \mathbb{R}$,

(d) the map $A$ is increasing on $[a_1, a_0] \cap \mathbb{R}$ and decreasing on $[b_0, b_1] \cap \mathbb{R}$,

(e) $a_1 \in \mathbb{R} \Rightarrow A(a_1) = 0$ and, similarly $b_1 \in \mathbb{R} \Rightarrow A(b_1) = 0$,

(f) $[a_0, b_0] \subseteq \text{Ker}(A)$.

Particular cases of fuzzy numbers are:

(i) The ordinary (or “non-fuzzy”) real numbers, where a real number $r \in \mathbb{R}$ is interpreted in the obvious way by means of its characteristic map $A_r : \mathbb{R} \to [0, 1]$ where $A_r(s) = 0$ if $r \neq s$ and $A_r(r) = 1$.

(ii) The triangular fuzzy numbers that are those for which $a_1, b_1 \in \mathbb{R}, a_0 = b_0, A(t a_1 + (1-t) a_0) = 1-t$ for every $t \in [0, 1]$ and also $A(t a_0 + (1-t) b_1) = t$ for every $t \in [0, 1]$. A triangular fuzzy number is said to be symmetric (or “isosceles”) if $a_0 = (a_1 + b_1)/2$. 

(iii) The polygonal fuzzy numbers of class \( n \) that are those for which \( a_1, b_1 \in \mathbb{R}, a_0 = b_0, \) and there exist real numbers \( c_1, \ldots, c_n \) and \( d_1, \ldots, d_n \) such that \( a_1 < c_1 < \cdots < c_n < a_0 = b_0 < d_n < d_{n-1} < \cdots < d_1 < b_1 \) satisfying that the function \( A \) is convex, piecewise linear, and \( A(c_i) = A(d_i) = i/(i + 1) \) \((i = 1, \ldots, n)\). A polygonal fuzzy number of class \( n \) is said to be symmetric if \( a_0 = (a_1 + b_1)/2 = (c_i + d_i)/2 \) \((i = 1, \ldots, n)\).

An example of this kind of fuzzy numbers is given by the map \( A : [-1, 1] \rightarrow [0, 1] \) defined as follows: (i) \( A(x) = (3x + 3)/2 \) \( x \in [-\frac{1}{2}, 0] \), (ii) \( A(x) = (x + 2)/2 \) \( x \in [-1, -\frac{1}{2}] \), (iii) \( A(x) = (-x + 2)/2 \) \( x \in [0, \frac{1}{2}] \), (iv) \( A(x) = (-3x + 3)/2 \) \( x \in [\frac{1}{2}, 1] \). In this case, \( a_1 = -1, c_1 = -\frac{2}{3}, c_2 = -\frac{1}{2}, a_0 = 0, d_2 = \frac{1}{2}, d_1 = \frac{2}{3}, b_1 = 1 \).

3. Orderings on fuzzy numbers and representations of binary relations

Despite it not being known how to endow the set of fuzzy numbers with a total order, if we restrict our attention to some special classes of fuzzy numbers, we can define suitable orderings (even, in some cases, total orders). As we shall see, we can get a wide range of “ordered” subsets of fuzzy numbers that can be used as suitable codomains to interpret binary relations on sets. Actually, this wide range of orderings is much richer than the usual real line \( \mathbb{R} \) with its Euclidean order \( \leq \).

To put an example let us define the following total order on the set \( ST \) of symmetric triangular fuzzy numbers: An element of \( ST \) can be identified by the numbers \( a_0 \) and \( a_1 \) that appear in its definition, so that we can denote an element of \( ST \) as \( \{a_0, a_1\} \). In consequence, we say that \( \{a_0, a_1\} \preceq \{a'_0, a'_1\} \) if and only if \( a_0 < a'_0 \) or \( a_0 = a'_0 \) and \( a_1 \leq a'_1 \).

The underlying idea of this ordering is very simple: A symmetric triangular fuzzy number can be interpreted as a “vague” manner of looking at the real number \( a_0 \), and the smallest is the difference \( a_0 - a_1 \) (i.e.: the biggest is \( a_0 \preceq a_0 \)), the “closest” is the fuzzy number to be an ordinary real number. Thus, to compare two symmetric triangular fuzzy numbers we first compare the real number (i.e., \( a_0 \) and \( a'_0 \)) that they intend to represent (in a vague, fuzzy manner), and in case of coincidence we compare then such “closedness” of that fuzzy numbers to be ordinary real numbers.

It is important to observe now that this totally ordered structure \( (ST, \preceq) \) is isotonic (order-isomorphic) to the subset \( (\mathcal{M}, \preceq_L) \) of the lexicographic plane \( (\mathbb{R}^2, \preceq_L) \), where \( \mathcal{M} = \{(x, y) \in \mathbb{R}^2, y \leq x\} \).

**Definition 2.** Given a nonempty set \( X \) endowed with a binary relation \( \mathcal{R} \), and a totally ordered set \( (S, \preceq_S) \) we say that \((X, \mathcal{R}) \) is representable in \( (S, \preceq_S) \) if there exists a map \( f : X \rightarrow S \) such that \( x \mathcal{R} y \iff f(x) \preceq_S f(y) \), for every \( x, y \in X \).

Let us see now, as an easy consequence of the above construction, that the whole lexicographic plane \( (\mathbb{R}^2, \preceq_L) \) and the set \( ST \) of symmetric triangular numbers endowed with this ordering \( \preceq \) are “interchangeable” or “equivalent” as codomains to represent total preorders on a set \( X \):

**Proposition 3.** The following assertions are equivalent for a set \( X \) endowed with a binary relation \( \mathcal{R} \):

(i) \((X, \mathcal{R}) \) is representable in \( (\mathbb{R}^2, \preceq_L) \),
(ii) \((X, \mathcal{R}) \) is representable in \( (ST, \preceq) \).
Proof. (i) ⇒ (ii): The map \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( g(x, y) = \left( \frac{3}{2} + \arctg x / \pi, \frac{1}{2} + \arctg y / \pi \right) \) for \((x, y) \in \mathbb{R}^2\) is a representation of \((\mathbb{R}^2, \preceq_L)\) in its totally ordered subset \(((1, 2) \times (0, 1), \preceq_L)\). Also, the map \( h : (1, 2) \times (0, 1) \to ST \) given by \( h(a_0, a_1) = \{a_0, a_1\} \in ST \) \((0 < a_1 < a_0 < 2)\) is a representation of \(((1, 2) \times (0, 1), \preceq_L)\) in \(ST\). Thus if \( f : X \to \mathbb{R}^2 \) is a representation of \((X, \mathcal{R})\) in \((\mathbb{R}^2, \preceq_L)\) it is plain that the composition \( h \circ g \circ f : X \to ST \) is a representation of \((X, \mathcal{R})\) in \((ST, \preceq)\).

(ii) ⇒ (i): By construction, \((ST, \preceq)\) can be represented in the obvious way in \((\mathcal{M}, \preceq_L)\). Let \( g \) such representation. Moreover, the embedding identity map \( i : \mathcal{M} \to \mathbb{R}^2 \) is a representation of \((\mathcal{M}, \preceq_L)\) in \((\mathbb{R}^2, \preceq_L)\). Thus if \( f : X \to \mathbb{R}^2 \) is a representation of \((X, \mathcal{R})\) in \((ST, \preceq)\) it is plain that the composition \( i \circ g \circ f : X \to \mathbb{R}^2 \) is a representation of \((X, \mathcal{R})\) in \((\mathbb{R}^2, \preceq_L)\).

Notice again that the lexicographic plane \((\mathbb{R}^2, \preceq_L)\) is not representable in the real line \((\mathbb{R}, \preceq)\) (see e.g. [28]). This is because if \( f : (\mathbb{R}^2, \preceq_L) \to (\mathbb{R}, \preceq) \) were a representation, then for every \( a \in \mathbb{R} \) it follows that \((f(a, 0), f(a, 1)) \subset \mathbb{R} \) is a nontrivial interval \( I_a \) of real numbers. Also if \( a \neq a' \) it follows that \( I_a \cap I_{a'} = \emptyset \). Hence there is an uncountable number of nontrivial nonintersecting intervals of real numbers. But this is impossible, since each nontrivial interval contains a rational number, and the set of rationals is countable.

Obviously, the real line can be represented in the lexicographic square. Since, as a codomain to represent binary relations, this set is equivalent to \((ST, \preceq)\), we see that the totally ordered set \((ST, \preceq)\) of symmetric triangular fuzzy numbers furnishes an example of a totally ordered structure that is richer than the Euclidean real line \((\mathbb{R}, \preceq)\).

Remark 4. It is straightforward to see that, if we consider the set \(\mathcal{SP}(n)\) of symmetric polygonal fuzzy numbers of class \(n\), and endow it with the total order \(\preceq\) given by \([a_1, c_1, \ldots, c_n, a_0, d_n, \ldots, d_1, b_1] \preceq [a_1', c_1', \ldots, c_n', a_0', d_n', \ldots, d_1', b_1']\) if and only if \(a_0 < a_0',\) or \(a_0 = a_0', a_1 < a_1',\) or there exist \(k \leq n\) such that \(a_0 = a_0', a_1 = a_1', c_i = c_i'\) \((i = 1, \ldots, k - 1),\) \(c_k < c_k',\) or \(a_0 = a_0', a_1 = a_1', c_i = c_i'\) \((i = 1, \ldots, n - 1),\) \(c_n < c_n',\) we obtain that \((\mathcal{SP}(n), \preceq)\) is order-isomorphic to a subset of the lexicographic space \((\mathbb{R}^{n+2}, \preceq_L)\) in a way that, similarly to Proposition 3, the following assertions are equivalent for a set \(X\) endowed with a binary relation \(\mathcal{R}\):

(i) \((X, \mathcal{R})\) is representable in \((\mathbb{R}^{n+2}, \preceq_L)\).
(ii) \((X, \mathcal{R})\) is representable in \((\mathcal{SP}(n), \preceq)\).

Remark 5. The lexicographic totally ordered spaces \((\mathbb{R}^d, \preceq_L)\) and \((\mathbb{R}^m, \preceq_L)\) are isotonic (order-isomorphic) if and only if \(n = m\). Actually, \((\mathbb{R}^n, \preceq_L)\) can be represented in \((\mathbb{R}^m, \preceq_L)\) if and only if \(n \leq m\).

That is, roughly speaking we can say that we obtain here a sort of “theorem of topological dimension” (let us remember that the Euclidean space \(\mathbb{R}^n\) is homeomorphic to \(\mathbb{R}^m\) if and only if \(n\) and \(m\) coincide), but now based on total orders of lexicographic type (and, as a matter of fact, similar to the aforementioned orderings on sets of polygonal fuzzy numbers). An equivalent result, attributed to A.M. Gleason, appears as an exercise in [11, p. 201, ex. 13].

Let us prove the assertion for the sake of completeness: Observe that it is enough to prove that \((\mathbb{R}^{n+1}, \preceq_L)\) cannot be represented in \((\mathbb{R}^n, \preceq_L)\) for any \(n \in \mathbb{N}\). The proof for \(n = 1\) has been already made in the paragraphs previous to Remark 4. Reasoning by induction, suppose now that for some \(n \geq 2,\) \(f : (\mathbb{R}^{n+1}, \preceq_L) \to (\mathbb{R}^n, \preceq_L)\) is a representation. Three cases may occur:
Case 1: If there exists \( a_1, a'_1 \in \mathbb{R} \) with \( a_1 < a'_1 \), and \( b_1 \in \mathbb{R} \) such that \( f(a_1, a_2, \ldots, a_{n+1}) = (b_1, b_2, \ldots, b_n) \) and also \( f(a'_1, a'_2, \ldots, a'_{n+1}) = (b_1, b'_2, \ldots, b'_n) \) for some \( a_2, a_3, \ldots, a_{n+1}, a'_2, \ldots, a'_{n+1}, b_2, \ldots, b_n, b'_2, \ldots, b'_n \in \mathbb{R} \), taking a value \( a \in (a_1, a'_1) \subset \mathbb{R} \) we have that \( f \) maps \( (a) \times \mathbb{R}^n \) into \( (b_1) \times \mathbb{R}^{n-1} \).

Since \( f \) is order-preserving, it would induce a representation of \((\mathbb{R}^n, \preceq_L)\) in \((\mathbb{R}^{n-1}, \preceq_L)\), which contradicts the induction hypothesis.

Case 2: If there exists \( a_1 \in \mathbb{R} \) such that, for every \( a_2, a_2, \ldots, a_{n+1}, a'_2, \ldots, a'_{n+1} \in \mathbb{R} \) with \( (a_1, a_2, \ldots, a_{n+1}) \preceq_L (a_1, a'_2, \ldots, a'_{n+1}) \) it never holds that \( f(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{R}^n \) and \( f(a_1, a'_2, \ldots, a'_{n+1}) \in \mathbb{R}^n \) start with the same coordinate, then for all \( \{a_1\} \times \mathbb{R}^n \), \( \pi_1 \) being the projection on the first coordinate, the composition map \( \pi_1 \circ f \) would furnish a representation of \((\mathbb{R}^n, \preceq_L)\) in \((\mathbb{R}, \leq)\), in contradiction with the induction hypothesis.

Case 3: If there exists \( i < n \) jointly with \( a_1, a_2, \ldots, a_{i-1}, a_i, a'_i \in \mathbb{R} \), and \( b_1, b_2, \ldots, b_i \in \mathbb{R} \) such that \( a_i < a'_i \), and \( f(a_1, a_2, \ldots, a_{i-1}, a_i, a'_{i+1}, \ldots, a_{n+1}) = (b_1, b_2, \ldots, b_i, b_{i+1}, \ldots, b_n) \) and also \( f(a_1, a_2, \ldots, a_{i-1}, a'_i, a'_{i+1}, \ldots, a'_{n+1}) = (b_1, b_2, \ldots, b_i, b'_{i+1}, \ldots, b'_n) \) for some \( a_{i+1}, \ldots, a_{n+1}, a'_{i+1}, \ldots, a'_{n+1}, b_{i+1}, \ldots, b'_n \in \mathbb{R} \), then \( f \) maps \( \{a_1\} \times \cdots \times \{a_{i-1}\} \times (a_i, a'_i) \times \mathbb{R}^{n-i} \) into \( \{b_1\} \times \cdots \times \{b_i\} \times \mathbb{R}^{n-i} \). Since \( f \) is order-preserving, it would induce a representation of \((a_i, a'_i) \times \mathbb{R}^{n-i}, \preceq_L)\) in \((\mathbb{R}^{n-i}, \preceq_L)\). But \((a_i, a'_i) \times \mathbb{R}^{n-i}, \preceq_L)\) is plainly isotonic to \((\mathbb{R}^{n-i+1}, \preceq_L)\), so that we get a representation of \((\mathbb{R}^{n-i+1}, \preceq_L)\) in \((\mathbb{R}^{n-i}, \preceq_L)\), which contradicts the induction hypothesis.

**Remark 6.** An alternative way to interpret the lexicographic space \((\mathbb{R}^n, \preceq_L)\) with \( n \geq 2 \) as a suitable ordered set of fuzzy numbers could be the use of *multidimensional fuzzy numbers*. The case of fuzzy numbers of dimension 2 may be seen in [34, p. 166 and ff]. Thus, to put an example, we can consider a *pyramidal (bidimensional) fuzzy number* as a map \( A = [x - e, x + e] \times [y - e, y + e] \subset \mathbb{R}^2 \to [0, 1] \), defined (for some \( e > 0 \)) as the one whose graph is the pyramid whose base is the square \([x - e, x + e] \times [y - e, y + e]\) in the plane \( z = 0 \), and whose summit is the point \((x, y, 1)\). Such a fuzzy number may be represented as \([x, y, e]\). Denote by \( \mathcal{P} = \{(x, y, e) : (x, y) \in \mathbb{R}^2, e > 0\} \) this set of pyramidal fuzzy numbers, and endow it with the total order \( \preceq \) given by \((x, y, e) \preceq (x', y', e')\) if and only if \((x, y, e) \preceq (x', y', e')\) in the lexicographic ordering of \(\mathbb{R}^3\). It is straightforward to get now the following result:

*The following assertions are equivalent for a set \( X \) endowed with a binary relation \( \mathcal{R} \):

(i) \((X, \mathcal{R})\) is representable in \((\mathbb{R}^3, \preceq_L)\),
(ii) \((X, \mathcal{R})\) is representable in \((\mathcal{P}, \preceq)\)."

(Obviously this result can be generalized to higher dimensions).

Also, another different alternative way to interpret the lexicographic space \((\mathbb{R}^3, \preceq_L)\) as an adequate set of fuzzy numbers consists in defining the following total order on the set \( T \) of triangular (but not necessarily symmetric!) *fuzzy numbers*: Let us denote an element of \( T \) as \((a_1, a_0, b_1)\). Endow \( T \) with the total order \( \preceq_T \) given by \( (a_1, a_0, b_1) \preceq_T (a'_1, a'_0, b'_1) \) if and only if \((a_0, (a_1 + b_1)/2, \arctan((b_1 - a_0)/(a_0 - a_1))) \preceq_L (a'_0, (a'_1 + b'_1)/2, \arctan((b'_1 - a'_0)/(a'_0 - a'_1))) \) in the lexicographic ordering \( \preceq_L \) of \(\mathbb{R}^3\). Indeed, it is straightforward to prove now the following fact:

*The following assertions are equivalent for a set \( X \) endowed with a binary relation \( \mathcal{R} \):

(i) \((X, \mathcal{R})\) is representable in \((\mathbb{R}^3, \preceq_L)\),
(ii) \((X, \mathcal{R})\) is representable in \((T, \preceq_T)\).*
(Obviously this result can be generalized to higher dimensions using suitable classes of non-symmetric polygonal fuzzy numbers).

4. Representations of interval orders and biorder correspondences through fuzzy numbers

Let us go back to the symmetric triangular fuzzy numbers. We can identify a symmetric triangular fuzzy number \([a_0, a_1] \in \mathcal{ST}\), where we know that \(a_0 = (a_1 + b_1)/2\) (or equivalently \(b_1 = 2a_0 - a_1\)), with the interval \([a_1, b_1]\) of real numbers, and, accordingly, define a new binary relation \(\preceq_I\) on \(\mathcal{ST}\) as follows: \(\{a_0, a_1\} \preceq_I \{a'_0, a'_1\}\) if and only if \(a_1 \leq b'_1 = 2a'_0 - a'_1\).

It is straightforward to see now that \(\preceq_I\) is an interval order on \(\mathcal{ST}\). It is plain that \(\preceq_I\) is not transitive. Obviously the structure \((\mathcal{ST}, \preceq_I)\) is representable as an interval order through the pair of functions \(F, G : \mathcal{ST} \rightarrow \mathbb{R}\) given by \(F([a_0, a_1]) = a_1; G([a_0, a_1]) = b_1 = 2a_0 - a_1\), for every \([a_0, a_1] \in \mathcal{ST}\). As a matter of fact, the next results follows immediately:

**Theorem 7.** The following assertions are equivalent for a set \(X\) endowed with an interval order \(\mathcal{R}\):

(i) \((X, \mathcal{R})\) is representable (as an interval order),

(ii) \((X, \mathcal{R})\) is representable in \((\mathcal{ST}, \preceq_I)\) in the sense that there exists a map \(H : X \rightarrow \mathcal{ST}\) such that \(x \mathcal{R} y \iff H(x) \preceq_I H(y)\), for every \(x, y \in X\).

**Proof.** (i) \(\Rightarrow\) (ii): Let \(F, G : X \rightarrow \mathbb{R}\) be a pair of functions that represent the interval order \(\mathcal{R}\) (i.e., \(x \mathcal{R} y \iff F(x) \leq G(y)\) \((x, y \in X)\)). Define now the map \(H : X \rightarrow \mathcal{ST}\) by \(H(x) = \{(F(x) + G(x))/2, F(x)\} \in \mathcal{ST}\) \((x \in X)\). It is now clear that \(H\) is a representation of \((X, \mathcal{R})\) in \((\mathcal{ST}, \preceq_I)\).

(ii) \(\Rightarrow\) (i): Conversely, let \(H : X \rightarrow \mathcal{ST}\) be a representation of \((X, \mathcal{R})\) in \((\mathcal{ST}, \preceq_I)\). Given \(x \in X\), let \(H(x) = \{a_0(x), a_1(x)\} \in \mathcal{ST}\). Let \(F, G : X \rightarrow \mathbb{R}\) be a pair of functions defined by \(F(x) = a_1(x)\) and \(G(x) = b_1(x) = 2a_0(x) - a_1(x)\) \((x \in X)\). It is clear now that \(F, G\) furnish a representation of \((X, \mathcal{R})\) as an interval order. \(\square\)

**Remark 8.** Observe also that if \(F, G\) furnish a representation of \((X, \mathcal{R})\) as an interval order, then \(G : X \rightarrow \mathbb{R}\) is a representation of the associated total preorder \(\mathcal{R}^*\), and similarly \(F : X \rightarrow \mathbb{R}\) is a representation for the associated total preorder \(\mathcal{R}^{**}\).

**Remark 9.** The representation of interval orders may still be handled by means of the representation of some adequate total preorder associated with the given interval order. Actually, there is a technique, introduced in a seminal paper by Doignon et al. [20], that allows us to represent correspondences from a set \(A\) to another set \(X\) instead of, just, binary relations defined on a set \(X\). Thus, we say that a correspondence \(\mathcal{C}\) from a nonempty set \(A\) to a nonempty set \(X\) such that \(A \cap X = \emptyset\) is representable by a pair of real-valued functions \(\{u, v\}\) with \(v : A \rightarrow \mathbb{R}\) and \(u : X \rightarrow \mathbb{R}\), if the following condition holds \(a\mathcal{C}x \iff v(a) < u(x)\) \((a \in A, x \in X)\). By technical reasons the sets \(A\) and \(X\) are taken to be disjoint. But this is not a severe restriction: If \(\mathcal{C}\) is a correspondence from \(A\) to \(X\) (with \(A \cap X \neq \emptyset\)) call \(A' = A \times \{0\}; X' = X \times \{1\}\) and consider the new correspondence \(\mathcal{C}'\) from \(A'\) to \(X'\) given by \((a, 0)\mathcal{C}'(x, 1) \iff a\mathcal{C}x\). Obviously \(A'\) and \(X'\) are now disjoint, whereas \(\mathcal{C}\) is representable if and only if \(\mathcal{C}'\) is. This allows us to consider, as a particular case, binary relations defined on a set \(X\).
In [20] a special kind of correspondence from \( A \) to \( X \) was introduced, namely, a biorder. The technique to represent a biorder \( C \) consists in defining an associated total preorder \( R \) on the disjoint union of \( A \) and \( X \). If \( a \in A \) and \( x \in X \) then \( aCx \iff aRx \). When the preorder \( R \) is representable through a utility function \( f \), the restrictions of \( f \) to \( A \) and \( X \) furnish two real-valued maps \( u : A \rightarrow \mathbb{R} \) and \( v : X \rightarrow \mathbb{R} \) such that the pair \((u, v)\) represents \( C \). These techniques can be used to find a set of conditions that characterize the representability of an interval order \( R \) defined on a set \( E \). Thus, a correspondence \( C \) from a set \( A \) to another set \( X \) is said to be a biorder if for every \( a, b \in A \) and \( x, y \in X \) the following condition holds: \( aCx \) and \( bCy \Rightarrow aCy \) or \( bCx \).

Notice that this concept of a biorder generalizes the concept of an interval order because, in the particular case \( A = X \), a reflexive biorder \( C \) defined on the set \( E = A = X \) is just an interval order. However, since in biorder theory it is much more usual to consider sets \( A \) and \( X \) taken to be disjoint, henceforward we shall consider two disjoint sets \( A \) and \( X \).

Let \( C \) be a correspondence from \( A \) to \( X \). Associated with \( C \), we consider the correspondence \( R \) from \( X \) to \( A \) given by \( xRa \iff \neg(aCx) (a \in A, x \in X) \) and the following binary relations \( P_1 \) on \( A \) and \( P_2 \) on \( X \): \( aP_1b \iff \) there exists \( x \in X \) such that \( aCx \land b \) and \( xP_2y \iff \) there exists \( a \in \) such that \( xRaCy (x, y \in X) \). Define also \( aP_1b \iff \neg(bP_1a) \iff (bCz \Rightarrow aCy \) for every \( z \in X \), \( (a, b \in A), xRaCy \iff \neg(yP_2x) \iff (cCx \Rightarrow cCy \) for every \( c \in A \), \( x, y \in X \). The case when the correspondence \( C \) is a biorder is characterized through the following fact.

**Proposition 10.** Let \( C \) be a correspondence from \( A \) to \( X \). Then the following statements are equivalent:

(i) \( C \) is a biorder,
(ii) \( R_1 \) is a total preorder on \( A \),
(iii) \( R_2 \) is a total preorder on \( X \).

**Proof.** Let us prove that (i) \( \Rightarrow \) (ii): By its own definition, \( R_1 \) is a reflexive and transitive binary relation. To see that it is also complete, let \( a, b \in A \) and assume by contradiction that \( \neg(aR_1b) \land \neg(bR_1a) \) hold. This is equivalent to \( bP_1a \land aP_1b \) so that there exist \( z, w \in X \) such that \( bCzR_1a \land aCwR_1b \). Since \( C \) is a biorder, \( bCz \) and \( aCw \) imply that either \( bCw \) or \( aCz \) holds. But both cases lead to a contradiction. Let us prove now that (ii) \( \Rightarrow \) (i): Let \( a, b \in A \) and \( x, y \in X \) such that \( aCx \land bCy \). Assume by contradiction that \( yRa \land xRb \) hold. Then we get \( bP_1a \land aP_1b \). But this is now a contradiction, because \( R_1 \) is a total preorder, so that \( P_1 \) is an asymmetric binary relation. Finally, observe that (i) \( \iff \) (iii) is entirely analogous to (i) \( \iff \) (ii). \( \square \)

Given a biorder \( C \) from \( A \) to \( X \), not only \( A \) and \( X \) are endowed with total preorders (respectively, \( R_1 \) and \( R_2 \)). Actually there is an extension of both such preorders to a total preorder \( R' \) defined on the disjoint union \( A \cup X \), as the next key fact shows (see [20]).

**Proposition 11.** Let \( C \) be a biorder from the set \( A \) to the set \( X \). Then \( A \cup X \) can be given a total preorder \( R' \), whose respective restrictions to \( A \) and \( X \) are \( R_1 \) and \( R_2 \), defined as follows: \( aR'_b \iff aR_1b \) for every \( a, b \in A \), \( aR'_x \iff (yRa \land xRb \Rightarrow yRb (b \in A, y \in X)) \) for every \( a \in A, x \in X \), \( xR'_a \iff xRa \) for every \( a \in A, x \in X \), \( xR'_y \iff xR_2y \) for every \( x, y \in X \).

**Proof.** See Proposition 1 in [13]. \( \square \)
Now it is straightforward to see that if the total preorder $\mathcal{R}'$ on $A \cup X$ is representable by means of an order-preserving real-valued utility function, the given biorder C is also a representable correspondence.

Using symmetric triangular fuzzy numbers, we also have the following easy equivalence:

**Proposition 12.** Let $\mathcal{C}$ be a biorder from the nonempty set $A$ to the nonempty set $X$, with $A \cap X = \emptyset$. The following assertions are equivalent:

(i) The biorder $\mathcal{C}$ is representable by a pair of real-valued functions $(u, v)$ with $v : A \rightarrow \mathbb{R}$ and $u : X \rightarrow \mathbb{R}$ (i.e., $a \mathcal{C} x \iff v(a) < u(x)$ $(a \in A, x \in X)$.)

(ii) $\mathcal{C}$ is representable in the set of fuzzy numbers $\mathcal{S}\mathcal{T}$, endowed with the interval order $\preceq_I$, in the sense that there exists a map $H : A \cup X \rightarrow \mathcal{S}\mathcal{T}$ such that $a \mathcal{C} x \iff \neg (H(x) \preceq_I H(a))$, for every $a \in A, x \in X$.

**Proof.** To prove that (i) implies (ii), notice that if $(u, v)$ represents $\mathcal{C}$ defining $H(a) = \{v(a) - 1, v(a) - 2\} \in \mathcal{S}\mathcal{T}$ $(a \in A)$ and $H(x) = \{u(x) + 1, u(x)\} \in \mathcal{S}\mathcal{T}$ $(x \in X)$ we have that $v(a) < u(x) \iff \neg (u(x) \preceq v(a)) \iff \neg (H(x) \preceq_I H(a))$. Conversely, to prove that (ii) implies (i) notice that if there exists a map $H : A \cup X \rightarrow \mathcal{S}\mathcal{T}$ such that $a \mathcal{C} x \iff \neg (H(x) \preceq_I H(a))$, for every $a \in A, x \in X$, being $H(y) = \{a_0(y), a_1(y)\} \in \mathcal{S}\mathcal{T}$ $(y \in A \cup X)$, we can define $v : A \rightarrow \mathbb{R}$ as $v(a) = b_1(a) = 2a_0(a) - a_1(a) (a \in A)$ and similarly define $u : X \rightarrow \mathbb{R}$ as $u(x) = a_1(x) (a \in A)$. Thus $a \mathcal{C} x \iff \neg (H(x) \preceq_I H(a)) \iff v(a) < u(x)$.

Finally, let us point out that if a biorder $\mathcal{C}$ is representable by a pair of real-valued functions $(u, v)$ with $v : A \rightarrow \mathbb{R}$ and $u : X \rightarrow \mathbb{R}$ then the map $u$ is a utility function that represents the total preorder $\mathcal{R}_2$ on $X$, and, similarly, $v$ is a representation for the total preorder $\mathcal{R}_1$ on $A$.

5. Representations of acyclic binary relations through fuzzy numbers

Consider now the following particular subset of $\mathcal{S}\mathcal{T}$, which we call the set of symmetric triangular fuzzy numbers of unitary base, which we denote by $\mathcal{S}\mathcal{T}\mathcal{U}$, and consists of all the elements $\{a_0, a_0 - \frac{1}{2}\} \in \mathcal{S}\mathcal{T}$. We endow $\mathcal{S}\mathcal{T}$ with the ordering $\preceq_I$, as above. We easily obtain the next result, similar to Theorem 7.

**Theorem 13.** The following assertions are equivalent for a set $X$ endowed with a semiorder $\mathcal{R}$ that is not a total preorder:

(i) $(X, \mathcal{R})$ is representable (as a semiorder) through a map $U : X \rightarrow \mathbb{R}$ and a strictly positive threshold $K > 0$, such that $x \mathcal{R} y \iff U(x) - U(y) \leq K$ $(x, y \in X)$,

(ii) $(X, \mathcal{R})$ is representable in $(\mathcal{S}\mathcal{T}\mathcal{U}, \preceq_I)$ in the sense that there exists a map $H : X \rightarrow \mathcal{S}\mathcal{T}\mathcal{U}$ such that $x \mathcal{R} y \iff H(x) \preceq_I H(y)$, for every $x, y \in X$.

**Proof.** (i) $\Rightarrow$ (ii): Let $U : X \rightarrow \mathbb{R}$ and $K > 0$ be a function and a threshold that represent the semiorder $\mathcal{R}$ (i.e., $x \mathcal{R} y \iff U(x) \leq U(y) + K (x, y \in X)$). Define now the map $H : X \rightarrow \mathcal{S}\mathcal{T}\mathcal{U}$ by $H(x) = \{U(x)/K, U(x)/K - \frac{1}{2}\} \in \mathcal{S}\mathcal{T}\mathcal{U}$ $(x \in X)$. It is plain that $H$ is a representation of $(X, \mathcal{R})$ in $(\mathcal{S}\mathcal{T}\mathcal{U}, \preceq_I)$. 


(ii) \(\Rightarrow\) (i): Conversely, let \(H : X \to STU\) be a representation of \((X, R)\) in \((STU, \lesssim)\). Given \(x \in X\), let \(H(x) = \{a_0(x), a_0(x) - \frac{1}{2}\} \in STU\). Observe now that, by definition, \(x R y \iff a_0(x) - \frac{1}{2} \leq 2a_0(y) - (a_0(y) - \frac{1}{2}) = a_0(y) + \frac{1}{2} \iff a_0(x) - a_0(y) \leq 1\). Thus, the function \(U : X \to \mathbb{R}\) given by \(U(x) = a_0(x) (x \in X)\), jointly with threshold 1, provides a representation of \(R\) as a semiorder. □

Given an interval order \(R\) defined on a nonempty set \(X\), its asymmetric part \(P\) is indeed transitive because by definition of an interval order we have that \(((x P y) \text{ and } (y P z))\) is equivalent to \(\iff (\neg(y R x) \text{ and } \neg(z R y))\) and this implies \(\neg(z R x) \text{ or } \neg(y R y))\). But now this implies \(\neg(z R x)\), or equivalently \(x P z\), by reflexivity of \(R\).

Consequently, \(P\) is an acyclic binary relation, because the existence of elements \(x_1, x_2, \ldots, x_n \in X\) such that \(x_i P x_{i+1} (i = 1, \ldots, n - 1)\) implies, by transitivity of \(P\), that \(x_1 P x_n\), so that it can never happen that \(x_n P x_1\) since \(P\) is asymmetric.

If the interval order \(R\) is representable through a pair of real-valued functions \(u, v : X \to \mathbb{R}\) such that \(x R y \iff u(x) \leq u(y) (x, y \in X)\), then we can also “represent” the binary relation \(P\) using such functions \(u\) and \(v\): notice that \(x P y \iff v(x) < u(y) (x, y \in X)\). The same idea can be used for representable semiorders. In that case \(x P y \iff u(x) + K < u(y) (x, y \in X)\), where \(u\) and \(K\) denote the real-valued function and the threshold that furnish the representation of the semiorder \(R\) as \(x R y \iff u(x) \leq u(y) + K (x, y \in X)\).

Thus, passing to the corresponding associated binary relation \(P\), an interval order (or a semiorder) \(R\) defined on a nonempty set \(X\) can be interpreted by means of an acyclic binary relation. Consequently, given an acyclic binary relation \(A\) on a nonempty set \(X\), we could try to find real-valued “representations” for \(A\) of the following kinds:

(a) Existence of two maps \(f, g : X \to \mathbb{R}\) such that \(x A y \iff f(x) < g(y) (x, y \in X)\) (see [48]).

(b) Existence of a map \(f : X \to \mathbb{R}\) and a nonnegative real constant \(k \geq 0\) or “threshold” such that \(x A y \iff f(x) + k < f(y) (x, y \in X)\). (Obviously, this is a particular case of (a).)

However, the existence of such kind of representations would carry severe restrictions on the binary relation \(A\). For instance, if \(A\) admits a representation through two maps \(f, g : X \to \mathbb{R}\) such that \(x A y \iff f(x) < g(y) (x, y \in X)\), the binary relation \(R\) defined by \(x R y \iff \neg(y A x) \iff \neg(x A y) \iff g(x) \leq f(y) (x, y \in X)\) has a representation of the kind introduced in [51]. Indeed, \(R\) is, a fortiori, an interval order, whereas not every acyclic binary relation has such property: Consider for instance \(X = \{1, 2, 3\}\) and define the binary relation \(A\) on \(X\) as follows: \(1 A 2; 2 A 3\). Let \(R = \neg(A)\) so that we have \(1 R 1; 1 R 2; 1 R 3; 2 R 2; 2 R 3; 3 R 1\) and \(3 R 3\). \(R\) is not an interval order because we have \(2 R 2\) and \(3 R 1\), but neither \(2 R 1\) nor \(3 R 2\).

There are some other possible “numerical representations” for a binary relation \(R\) on a nonempty set \(X\), which imply that \(R\) is acyclic.

A new example is a kind of representation introduced in [27], where a binary relation \(A\) is “representable” if there is a set-valued correspondence \(S\) from \(X\) into \(\mathbb{R}\) such that \(S(x)\) is a nonempty bounded subset, for every \(x \in X\), and it holds that \(x A y \iff S(x) \subset S(y)\), and, in addition, \(\sup S(x) < \sup S(y) (x, y \in X)\). Notice that this kind of representation can be reinterpreted by replacing the sets \(S(x) (x \in X)\) by their convex hulls \(\bar{S}(x)\), which, \(S(x)\) being bounded, must forcibly be intervals of real numbers \(\bar{S}(x) = [u(x), v(x)]\) (that may eventually collapse to a single point, that is, it may happen that \(u(x) = v(x)\) for some \(x \in X\)). It is plain that \(\sup \bar{S}(x) = v(x)\). Consequently, we have that \(x A y\) if and
only if \( u(y) \leq u(x) \leq v(x) < v(y) \) \((x, y \in X)\), and it is clear that this forces \( \mathcal{A} \) to be an acyclic binary relation on \( X \).

As a matter of fact, this kind of representation (and in general, any “representation” involving intervals of real numbers) can be reinterpreted again using symmetric triangular fuzzy numbers:

In this case, we define a new binary relation \( \tilde{A} \) on \( ST \), by declaring that \( \{a_0, a_1\} \tilde{\leq} \{a'_0, a'_1\} \iff a'_1 \leq a_1 \leq a_0 \leq a'_0 \) \((\{a_0, a_1\}, \{a'_0, a'_1\} \in ST)\). Accordingly, we say that a binary relation \( \mathcal{A} \) is representable (“in the sense of Fishburn”, so to say) if there exists a map \( F : X \rightarrow ST \) such that \( x \mathcal{A} y \iff F(x) \tilde{\leq} F(y) \) \((x, y \in X)\).

Unfortunately, despite several kinds of numerical representations being built to deal with some particular classes of acyclic binary relations (as, e.g., the asymmetric parts of interval orders), there is no “universal” numerical representation for acyclic binary relation. Perhaps, as pointed out in [3], the best way to find numerical “representations” for acyclic binary relations is by means of weak utilities. Given a binary relation \( \mathcal{A} \) defined on a nonempty set \( X \), a weak utility for \( \mathcal{A} \) is a map \( F : X \rightarrow \mathbb{R} \) such that \( x \mathcal{A} y \Rightarrow F(x) < F(y) \) \((x, y \in X)\). Observe that a weak utility is not a “representation” in the sense that the fact \( F(x) < F(y) \) only provides a necessary (but not always sufficient!) condition to be \( x \mathcal{A} y \) \((x, y \in X)\). In other words, dealing with weak utilities we have an implication (“if”), but not an equivalence (“if and only if”). Accordingly, we shall say that \( \mathcal{A} \) is weakly representable in the Euclidean real line \((\mathbb{R}, \leq)\) if there exists a weak utility for \( \mathcal{A} \).

Thus, we can say that acyclicity is the natural requirement to study weak utilities, and weak utilities are the most natural way to “represent” acyclic binary relations, for two main reasons: firstly because the existence of a weak utility for a binary relation \( \mathcal{A} \) forces \( \mathcal{A} \) to be acyclic, and secondly because acyclicity and existence of weak utilities are equivalent in countable spaces, as proved in [15].

**Remark 14.** The existence of weak utilities could be generalized to the existence of many other “weak representations” whose codomain is a set of fuzzy numbers (instead of just ordinary real numbers). For instance, we could say that a binary relation \( \mathcal{A} \) defined on a nonempty set \( X \) is weakly representable in the totally ordered set of symmetric triangular fuzzy numbers \((ST, \preceq)\) if there exists a map \( F : X \rightarrow ST \) such that \( x \mathcal{A} y \Rightarrow \neg(F(y) \preceq F(x)) \) \((x, y \in X)\).

It is straightforward to see now that if there exists a weak representation for \( \mathcal{A} \) in \((ST, \preceq)\), then the binary relation \( \mathcal{A} \) must be acyclic.

Since \((ST, \preceq)\) is equivalent to a totally ordered subset \((\mathcal{M}, \preceq_L)\) of the lexicographic plane \((\mathbb{R}^2, \preceq_L)\) which is not representable in the real line \((\mathbb{R}, \leq)\), and \((\mathbb{R}, \leq)\) is representable in \((\mathcal{M}, \preceq_L)\), we conclude that weak representations in \((ST, \preceq)\) are more general than weak representations in \((\mathbb{R}, \leq)\). For instance, as in the comments made just after Proposition 3, it can be seen that the asymmetric part \(<_L\) of lexicographic totally ordered plane \((\mathbb{R}^2, \preceq_L)\) does not admit a weak representation in the Euclidean real line \((\mathbb{R}, \leq)\).

6. Other kinds of representations of binary relations through fuzzy numbers

To complete the panorama, let us analyse the possibility of interpreting some other kinds of representations of binary relations by means of fuzzy numbers (or, at least, fuzzy sets).

Among the different kinds of representations that have been mentioned in the Introduction, there are some of them that need the set \( X \) to have some kind of extra structure, and, in addition, are particular cases
of some other well-known representations. To put an example, this happens with the *representability in the sense of Levin* [36], where a binary relation $\mathcal{R}$ on a nonempty set $X$ is said to be representable if there exists a measure $\mu$ defined on $X$, and a set-valued correspondence $F$ from $X$ into $X$ such that for every $x \in X$ the set $F(x)$ is $\mu$-measurable, and also $x \mathcal{R} y \iff \mu(F(x)) \leq \mu(F(y))$ $(x, y \in X)$, where $\mu(F(x))$ must be a real number (i.e., finite) for every $x \in X$: we observe that defining the real-valued map $U : X \rightarrow \mathbb{R}$ by $U(x) = \mu(F(x))$ $(x \in X)$, $U$ becomes a utility function that represents $\mathcal{R}$ and, in particular, $\mathcal{R}$ must actually be a total preorder.

Since there is an *arithmetic* on fuzzy numbers (see e.g. [34]), so that in particular they can be added, we could think on a generalization of Levin’s representations for binary relations through some kind of *measure whose codomain is a suitable set of fuzzy numbers*. Thus, for instance, the addition of fuzzy numbers behaves well when restricted to the subset $\mathcal{ST}$ of symmetric triangular fuzzy numbers (i.e., the sum of two elements of $\mathcal{ST}$ also belongs to $\mathcal{ST}$). However, to deal with concepts as $\sigma$-finite measure with values in $\mathcal{ST}$ we should also endow $\mathcal{ST}$ with some Hausdorff topology (and its corresponding concept of a *limit*) in order to deal with *series* of symmetric triangular fuzzy numbers understood as limits of sums of symmetric triangular fuzzy numbers, such that the number of elements to be added tends to infinity. As far as we know this approach would be new in the literature of representation of binary relations defined on a nonempty set.

A similar situation appears if we look for a kind representation as the ones studied in [7]. Now we say that a binary relation $\mathcal{R}$ defined on a nonempty set $X$ is *representable by a metric* if there exist a metric $d$ on $X$ and a subset $C \subset X$ such that $x \mathcal{R} y \iff d(x, C) \leq d(y, C)$ $(x, y \in X)$. Observe again that defining $U : X \rightarrow \mathbb{R}$ by $U(x) = d(x, C)$ $(x \in X)$, $U$ becomes a utility function and $\mathcal{R}$ is a total preorder.

We could also try to generalize the concept of metric, by considering a map $D$ defined on $X \times X$ and taking values in the totally ordered set $\mathcal{ST}$ of symmetric triangular fuzzy numbers, satisfying the following conditions for every $x, y, z \in X$:

(a) $0 \preceq D(x, y)$, where $\bar{0}$ stands for the ordinary real number $0$, but considered as an element of $\mathcal{ST}$.
(b) $D(x, y) = D(y, x)$.
(c) $D(x, y) = \bar{0} \iff x = y$.
(d) $D(x, z) \preceq D(x, y) \oplus D(y, z)$, where $\oplus$ stands for the addition of fuzzy numbers, as defined in [34, p. 14 and ff].

In this case, given $x \in X$ and $C \subset X$, the concept $D(x, C)$ needs definition, and it can be understood as the smallest element with respect to the total order $\preceq$ in $\mathcal{ST}$ of the subset $\{ [a_0, a_1] \in \mathcal{ST} : D(x, y) = [a_0, a_1] \}$ for some $y \in C$. This smallest element $D(x, C)$ must be postulated to exist and be unique, in order to give sense to this new kind of representation.

Again, as far as we know this generalization has not been explored in the literature concerning representations of binary relations or preference orderings.

At this point, we can also point out that dealing with the classical concept of a *metric* $d : X \times X \rightarrow \mathbb{R}$, it is well known that the map $d' : X \times X \rightarrow \mathbb{R}$ defined by $d'(x, y) = d(x, y)/(1 + d(x, y))$ $(x, y \in X)$ provides a new metric, equivalent to $d$, such that $d'$ takes values in $[0, 1]$. If we define the map $d^* : X \times X \rightarrow [0, 1]$ as $d^*(x, y) = 1 - d'(x, y)$ $(x, y \in X)$, it follows that if a binary relation $\mathcal{R}$ is representable by the metric $d$, then $x \mathcal{R} y \iff d(x, C) \leq d(y, C) \iff d'(x, C) \leq d'(y, C) \iff D_C^*(y) \leq D_C^*(x)$ $(x, y \in X)$, where the function $D_C^* : X \rightarrow [0, 1]$ is defined by $D_C^*(x) = 1 - d'(x, C)$ $(x \in X)$. 


The map $d^*$ can be interpreted as a suitable fuzzy set of the Cartesian product $X \times X$, so that the universe is $X \times X$, the support $\text{Supp}(d^*)$ coincides with $X \times X$, and the kernel $\text{Ker}(d^*)$ is the diagonal $\Delta = \{(x, x) : x \in X\}$ of $X \times X$. This fuzzy set could give us a vague or “fuzzy” idea of equality between the elements of the set $X$, since $d^*(x, y) = 1 \iff x = y (x, y \in X)$. Moreover, given a subset $C \subset X$, the map $D^*_C : X \to (0, 1]$ also defines a fuzzy set, whose universe is now $X$, that can be interpreted as a “fuzzy enlargement” of the set $C$. Notice, in particular, that $C \subset \text{Ker}(D^*(C))$.

Let us analyse now the kind of representation introduced in [47]: We say that a binary relation $\mathcal{R}$ defined on a nonempty set $X$ is representable by means of a hemisymmetric form if there is a map $H : X \times X \to \mathbb{R}$ such that $x \mathcal{R} y \iff H(x, y) \geq 0$ and, in addition, $H(x, y) + H(y, x) = 0 (x, y \in X)$. Observe, in particular that $H(x, x) = 0$ for every $x \in X$, so that $\mathcal{R}$ must be reflexive. The fact $H(x, y) = -H(y, x) (x, y \in X)$ implies that $\mathcal{R}$ is a complete binary relation because either $H(x, y) \geq 0$ or $H(y, x) \geq 0$ must hold. If $\mathcal{R}$ is representable through a hemisymmetric form, $H$, and we define the binary relation $\mathcal{P}$ on $X$ as $x \mathcal{P} y \iff -H(y, x) (x, y \in X)$, it follows that $x \mathcal{P} y \iff H(y, x) > 0$. Consequently, $H(y, x) < 0$, so that this new associated binary relation $\mathcal{P}$ is asymmetric. However, it may fail to be transitive, as we shall see later.

It is important to say that this kind of representation introduced in [47] is quite general. Thus, if we have a total preorder $\mathcal{R}$ on $X$, and $\mathcal{R}$ is representable by means of a utility function $F : X \to \mathbb{R}$ (i.e., $x \mathcal{R} y \iff F(x) \leq F(y)$), then the map $H : X \times X \to \mathbb{R}$ given by $H(x, y) = F(y) - F(x) (x, y \in X)$ is hemisymmetric and represents $\mathcal{R}$. Notice also that, in this case $H$ also verifies the following functional equation $H(x, y) + H(y, z) = H(x, z) (x, y, z \in X)$. But it is not true, in general, that every hemisymmetric form satisfies such an equation: For instance, consider the set $X = \{1, 2, 3\}$ and declare $1 \mathcal{R} 1; 1 \mathcal{R} 2; 2 \mathcal{R} 2; 2 \mathcal{R} 3; 3 \mathcal{R} 1; 3 \mathcal{R} 3$. Now consider $H : X \times X \to \mathbb{R}$ defined by $H(1, 2) = H(2, 3) = H(3, 1) = 1; H(1, 3) = H(2, 1) = H(3, 2) = -1; H(1, 1) = H(2, 2) = H(3, 3) = 0$. $H$ is hemisymmetric and represents $\mathcal{R}$. However, $H(1, 2) + H(2, 3) \neq H(1, 3)$. The associated binary relation $\mathcal{P}$ is given by $1 \mathcal{P} 2; 2 \mathcal{P} 3$ and $3 \mathcal{P} 1$, so that it is asymmetric, but not transitive. Observe also that $\mathcal{R}$ is not a total preorder, since it is not transitive because we have $1 \mathcal{P} 2, 2 \mathcal{P} 3$ but not $1 \mathcal{P} 3$.

This representation suggested by Shafer [47] can still generalized a bit more. Thus, in a paper by Riguet [45], a binary relation $\mathcal{R}$ on a nonempty set $X$ is said to be representable if there exists a bivariate map $F : X \times X \to \mathbb{R}$ such that $x \mathcal{R} y \iff F(x, y) \geq 0 (x, y \in X)$. Obviously, the representation of Shafer is the particular case of Riguet’s one, where the bivariate map $F$ is hemisymmetric.

However, the representation suggested by Riguet is trivial. That is, unless we ask the bivariate map $F$ to satisfy some particular restrictions that could correspond to some particular families of binary relations to be analysed, it happens that every binary relation admits a Riguet’s representation. (Just define $F : X \times X \to \mathbb{R}$ as $F(x, y) = 1$ whenever $x \mathcal{R} y$ and $F(x, y) = -1$ otherwise ($x, y \in X$).)

A further generalization was issued in a very old and pioneer work by Fechner [23], and also in some recent approach (e.g. Agaev and Aleskerov [1]): Now a binary relation $\mathcal{R}$ on a nonempty set $X$ is called representable if there exist a map $f : X \to \mathbb{R}$ and a suitable bivariate map $F : X \times X \to \mathbb{R}$, such that $x \mathcal{R} y \iff f(x) \leq f(y) + F(x, y) (x, y \in X)$: the representation suggested in [45] is the particular case of this last one where $f$ is the null map.

Again, this kind of representation becomes trivial unless we ask the bivariate map $F$ to accomplish, a priori, some additional conditions that could be characteristic of some families of binary relations to be represented.

As an important example of this situation, we see that Fechner’s representation generalizes the classical representations for interval orders (introduced in [51]) by means of two real-valued maps, so that an
interval order $\mathcal{R}$ on a nonempty set $X$ is representable if there exist $f, g : X \rightarrow \mathbb{R}$, such that $x \mathcal{R} y \iff f(x) \leq g(y)$ ($x, y \in X$). To see this, just define $F : X \times X \rightarrow \mathbb{R}$ as $F(x, y) = g(y) - f(y)$ ($x, y \in X$). If in addition $g(y) - f(y) = k > 0$ for every $y \in X$, where $k$ is a positive constant or “threshold”, we arrive at the kind of representation introduced in [5], which is the typical one to deal with semiorders that are not total preorders, as we have studied above.

A further final generalization could appear if we consider maps (even bivariate) that take values in, say, the totally ordered set of symmetric triangular numbers $(\mathcal{ST}, \preceq)$ instead of the Euclidean real line $(\mathbb{R}, \leq)$, so that a binary relation $\mathcal{R}$ on a nonempty set $X$ is representable if there exist a map $f : X \rightarrow \mathcal{ST}$, and a suitable bivariate map $F : X \times X \rightarrow \mathcal{ST}$, such that $x \mathcal{R} y \iff f(x) \preceq f(y) + F(x, y)$ ($x, y \in X$). Once more, as far as we know this generalization has not been explored in the literature. But, as in the case of Riguet’s and Fechner’s representations, it would become trivial unless we ask the map $f$ or the bivariate map $F$ to satisfy some additional conditions.

7. Conclusion

Having in mind applications in economics and social choice, we have shown that the concept of a fuzzy number is crucial to build models to represent a wide range of binary relations or “preferences” that could be defined on a set. Thus, the use of different families of fuzzy numbers as suitable codomains to represent binary relations introduces a new unifying tool to translate qualitative scales, given by binary relations that compare elements of a set, into quantitative numerical scales based on particular sets of fuzzy numbers, that we also endow with adequate orderings that depend on the class of binary relation to be analysed. Particular interesting cases, namely total preorders, interval orders, semiorders and acyclic binary relations (among others), have been shown to carry their characteristic theorems of representability based on some appropriate sets of fuzzy numbers.

Acknowledgements

Thanks are given to two anonymous referees for their valuable suggestions and comments.

References