Technical Note

INFORMATION FUNCTIONS & TEST CHARACTERISTIC CURVES FOR THE GRADED RESPONSE MODEL

Measured Progress

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Purpose

The purpose of this document is to provide the technical details on calculating information functions using Samejima's Graded Response Models as well as the calculation of the test characteristic function. We begin with some basic definitions, so that a consistent notation can be used throughout this document.

Overview of Graded Response Model

Samejima [1969, see also Samejima (1974)] defines the logistic form of the graded response model as:

$$P_{jk}(\theta) = \frac{\exp\left[Da_j\left(\theta - b_j + c_k\right)\right]}{1 + \exp\left[Da_j\left(\theta - b_j + c_k\right)\right]} - \frac{\exp\left[Da_j\left(\theta - b_j + c_{k+1}\right)\right]}{1 + \exp\left[Da_j\left(\theta - b_j + c_{k+1}\right)\right]}, \qquad (1)$$

where j indexes the items (j=1,2,3, … , n),
k represents a particular score category,
a represents item discrimination,
b represents the item location parameter,
c is the category parameter,
θ is the latent trait of interest, and
D is the 1.701 normalizing constant.

Equation 1 can be reparameterized such that:

$$P_{jk}(\theta) = P_{jk}^{+}(\theta) - P_{j,k+1}^{+}(\theta), \qquad (2)$$

where, for example, $P_{jk}^{+}(\theta)$ is simply the left term in Equation 1:

$$P_{jk}^{+}(\theta) = \frac{\exp\left[Da_j\left(\theta - b_j + c_k\right)\right]}{1 + \exp\left[Da_j\left(\theta - b_j + c_k\right)\right]} \qquad (3)$$

Equation 2 can be thought of as the probability of reponding according to category *k* (given a specific θ) is equal to the difference between responding in category *k* and in the next *k+1* category, with the assumption that:

$$P^{+}_{j0}(\theta) = 1$$

and

$$P^{+}_{j,m+1}(\theta) = 0\,,$$

where *m* is equal to the number of score points minus one. This then allows for the following set of important relationships to exist:

$$\begin{aligned}
P_{j0}(\theta) &= P^{+}_{j0}(\theta) - P^{+}_{j1}(\theta) &&= 1 - P^{+}_{j1}(\theta) \\
P_{j1}(\theta) &= P^{+}_{j1}(\theta) - P^{+}_{j2}(\theta) && \\
&\vdots && \\
P_{jm}(\theta) &= P^{+}_{jm}(\theta) - P^{+}_{j,m+1}(\theta) &&= P^{+}_{jm}(\theta).
\end{aligned}$$

Test Characteristic Curve

Using the formulations above we can now define the calculation of the test characteristic curve within the framework of the graded response model. Let us first begin by defining an *integer scoring function* (ISF). This function consists of a series of arbitrary scalars such that the values are consecutive and there are *m* unique values. For example, for a four point constructed response item (where scores range from 0 to 4, inclusive) our ISF might be: 1, 2, 3, 4.

Using this ISF we can now find the expected score for a given polytomous item as:

$$E_j(\theta) = \sum_{k=1}^{m} kP_{jk} = P_{j1}(\theta) + 2P_{j2}(\theta) + 3P_{j3}(\theta) \cdots mP_{jm}(\theta). \qquad (4)$$

In the case of a 3 point item Equation 4 can be constructed using the P^{+} notation [leaving out the (θ) for the sake of simplicity]:

$$\begin{aligned} E_j(\theta) &= \left[P_{1j}^{+} - P_{2j}^{+}\right] + 2\left[P_{2j}^{+} - P_{3j}^{+}\right] + 3\left[P_{3j}^{+}\right] \\ &= P_{1j}^{+} + P_{2j}^{+} + P_{3j}^{+}. \end{aligned} \tag{5}$$

In this 3 point item example, we can now define Equation 5 in terms of the familiary left hand terms of Equation 1 (leaving off *j* level subscripts):

$$\begin{aligned} E(\theta) = &\frac{\exp Da(\theta - b + c_1)}{1 + \exp Da(\theta - b + c_1)} + \frac{\exp Da(\theta - b + c_2)}{1 + \exp Da(\theta - b + c_2)} \\ &+ \frac{\exp Da(\theta - b + c_3)}{1 + \exp Da(\theta - b + c_3)} \end{aligned} \tag{6}$$

Equation 6 can be used in the case of when the response categories range from 0 to $m+1$ and include all possible values between 0 and $m+1$. In the case where a score category is not used then additional derivations are neccessary, but the same same scoring function method used in Equation 4 can still be employed. For example, in a 4 point item where no student achieved a score of 1 the observed scores are: 0, 2, 3, 4. Using Equation 4 we find that:

$$E_j(\theta) = 2P_{j2}(\theta) + 3P_{j3}(\theta) + 4P_{j4}(\theta). \tag{7}$$

Now expanding Equation 7 in a similar manner as we have seen in Equation 5 we obtain:

$$\begin{aligned} E_j(\theta) &= 2\left[P_{2j}^{+} - P_{3j}^{+}\right] + 3\left[P_{3j}^{+} - P_{4j}^{+}\right] + 4\left[P_{4j}^{+}\right] \\ &= 2P_{2j}^{+} + P_{3j}^{+} + P_{4j}^{+} \end{aligned} \tag{8}$$

Equation 8 reveals that it is necessary to "double up" the P_{2j}^{+} term to account for the missing "1" score point. This gaurantees that the test characteristic curve will asymptotically approach the total number of points as θ approaches $+\infty$. The expected scores obtained in Equations 8 and/or 5 can be summed across all polytomous items in the construction of the test characteristic curve.

The Information Function

As indicated by Muraki & Bock (1996), Samejima (1974) defined information for polytomous item response theory models as:

$$I_j(\theta) = \sum_{k=0}^{m_j} A_{jk}(\theta), \tag{9}$$

where

$$\sum_{k=0}^{m_j} A_{jk}(\theta) = \sum_{k=0}^{m_j} \frac{\left[\frac{\partial}{\partial\theta} P_{jk}(\theta)\right]^2}{P_{jk}(\theta)}. \tag{10}$$

For the logistic form of the graded response model the functional element in Equation 9 can be defined algebraically as:

$$A_{jk}(\theta) = D^2 a_j^2 \frac{\left\{P_{jk}^{+}(\theta)\left[1 - P_{jk}^{+}(\theta)\right] - P_{j,k+1}^{+}(\theta)\left[1 - P_{j,k+1}^{+}(\theta)\right]\right\}^2}{P_{jk}(\theta)}, \tag{11}$$

where all terms have been previsously defined [see also Baker (1992, pp. 239-243) for additional derivations]. Naturally, it follow that the overall test information function can be defined as the simple summation of Equation 9 across all *n* items:

$$I(\theta) = \sum_{j=1}^{n} I_j(\theta) \tag{12}$$

Following the usual convention within the dichotomous IRT framework (e.g., see Hambleton & Swaminathan, 1985) the standard error function is a simple derivation of Equation 12:

$$SE(\theta) = \left[\sum_{j=1}^{n} I_j(\theta)\right]^{-1/2} = \frac{1}{\sqrt{I(\theta)}}. \tag{13}$$

References

Baker, F. B. (1992). *Item Response Theory: Parameter Estimation Techniques*. New York: Marcel Dekker, Inc.

Muraki, E., & Bock, R. D. (1996). *PARSCALE: IRT Based Test Scoring and Item Analysis for Graded Open-ended Exercises and Performance Tasks* [computer program]. Chicago, IL: Scientific Software, Inc.

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